Resonance chains on Schottky surfaces

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Resonances – a quantum dynamical motivation

$H$: Hamilton operator of single particle open quantum system
Resonances – a quantum dynamical motivation

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What are resonances?
Resonances – a quantum dynamical motivation

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Poles of the meromorphically continued resolvent $R(z) = \frac{1}{iH+z}$
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Laplace transformation of time evolution operator:

$$\int_0^\infty e^{-iHt} e^{-zt} dt = \left. \frac{e^{(-iH-z)t}}{-iH - z} \right|_0^\infty = \frac{1}{iH + z} = R(z)$$
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$$U(t) = \frac{1}{2\pi i} \int_\gamma R(z) e^{zt} dz$$
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$$= \sum_{\lambda_i} e^{\lambda_i t} \Pi_\lambda + O(e^{-Ct})$$
Schottky surfaces

Def. Schottky surface:

- non-compact surface of constant negative curvature ($X = \Gamma \backslash \mathbb{H}^2$)
- finite genus
- finite number of funnels
- no cusps
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Hamiltonian: \(H = -\Delta_X \leftrightarrow \text{“free quantum particle”}\)
**Schottky surfaces**

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Hamiltonian: \( H = -\Delta_X \leftrightarrow \) “free quantum particle”
- Neg. curvature \( \Rightarrow \) geodesic flow is chaotic
- \( \Rightarrow \) Model for open quantum chaotic system

(Relations to number theory)
Resonances

\( X \) has infinite volume \( \Rightarrow \)

\[
\sigma_{L^2}^{\text{disc}}(\Delta X) \subset (0, 1/4) \text{ finite}
\]

\[
\sigma_{L^2}^{\text{cont}}(\Delta X) = [1/4, \infty)
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resolvent:

$$R_X(s) := (\Delta X - s(1 - s))^{-1} : L^2 \to L^2$$
Resonances

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resolvent:

\[ R_X(s) := (\Delta X - s(1 - s))^{-1} : L^2_{\text{comp}} \to L^2_{\text{loc}} \]

$R_X(s)$ is meromorphic family of operators with poles of finite multiplicity (Mazzeo-Melrose 86)

\[ \text{Res}(X) := \{\text{poles of } R_X(s)\} \]
Fractal Weyl law

Spectral Geometry:

Spectral properties $\leftrightarrow$ Geometry
Fractal Weyl law

Spectral Geometry:

Spectral properties $\leftrightarrow$ Geometry

Let $C_0 \in \mathbb{R}$

$$N_{C_0}(k) := \# \{ s \in \text{Res}(X), \text{Re}(s) > C_0, |\text{Im}(s)| < k \}$$
Fractal Weyl law

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then there is $C' > 0$ with

$$N_{C_0}(k) \leq C'k^{1+\delta}$$

(Zworski, 1999 Invent.Math.)
Spectral Geometry:

**Spectral properties ↔ Geometry**

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$$N_{C_0}(k) := \# \{ s \in \text{Res}(X), \text{Re}(s) > C_0, |\text{Im}(s)| < k \}$$

then there is $C' > 0$ with

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(Zworski, 1999 Invent.Math.)

Remark: $\delta_{\Gamma} = \dim_H(\Lambda_{\Gamma})$

**Conjecture: Exponent is sharp**
Theorem (Naud 2005)

*There is* $\varepsilon > 0$, *such that*

$$\# \{ s \in \text{Res}(X), \text{Re}(s) > \delta_{\Gamma} - \varepsilon \} < \infty$$
Theorem (Naud 2005)

There is $\varepsilon > 0$, such that

$$\# \{ s \in \text{Res}(X), \Re(s) > \delta_\Gamma - \varepsilon \} < \infty$$
Spectral Gap

**Theorem (Naud 2005)**

There is $\varepsilon > 0$, such that

$$\# \{ s \in \text{Res}(X), \text{Re}(s) > \delta_\Gamma - \varepsilon \} < \infty$$

**Conjecture:** $\varepsilon = \delta_\Gamma / 2$
Numerical Tests of Conjectures

Borthwick 2014 Exp.Math.

Fractal Weyl Conjecture (✓)
Numerical Tests of Conjectures

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- Fractal Weyl Conjecture (✓)
- Gap Conjecture (✗)
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- Fractal Weyl Conjecture (√)
- Gap Conjecture (✗)
- Observation of resonance chains:
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- Observation of resonance chains:
Resonance chains in physics

3-disk system:

microdisk cavity/laser (Main Wiersig 2008):

Ruelle-Resonances (Gaspard-Ramires 1992):
Questions

1) On which surfaces do we see chains?
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2) Approximative formula for these chains?
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Concerning 1):
S. Barkhofen, F. Faure, T.W., 2014

Resonance chains ↔ Clustering of length spectrum
Resonance Chains

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Concerning 1):
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Resonance chains $\Leftrightarrow$ Clustering of length spectrum

Concerning 2):

$X_{l_1,l_2,l_3}$
Resonance Chains

Questions

1) On which surfaces do we see chains?
2) Approximative formula for these chains?

Concerning 1):
S. Barkhofen, F. Faure, T.W., 2014

Resonance chains $\leftrightarrow$ Clustering of length spectrum

Concerning 2):

\[ X_{n_1 \ell, n_2 \ell, n_3 \ell} \]

\[ X_{l_1, l_2, l_3} \]

\[ \Rightarrow \text{study } X_{n_1 \ell, n_2 \ell, n_3 \ell} \]

in the limit $\ell \to \infty$
Rescaled resonances

\[ \text{Res}(X_{4\ell,4\ell,5\ell}) \text{ for } \ell = 3 \text{ and } 4 \]
Rescaled resonances

\[ \ell \cdot \text{Res}(X_{4\ell, 4\ell, 5\ell}) \text{ for } \ell = 3 \text{ and } 4 \]
Theorem (T.W., 2015)

Suppose \( n_i + n_j > n_k \) and define

\[
P_{n_1,n_2,n_3}(x) := 1 - 2(x^{n_1} + x^{n_2} + x^{n_3}) + x^{2n_1} + x^{2n_2} + x^{2n_3} \\
+ 2(x^{n_1+n_2} + x^{n_2+n_3} + x^{n_1+n_3}) - 4x^{n_1+n_2+n_3}.
\]

Then “an arbitrary large fixed number of rescaled resonances converges in the limit \( \ell \to \infty \) uniformly to the zeros of \( P_{n_1,n_2,n_3}(e^{-s}) \)”. 
\( \ell \cdot \text{Res}(X_{4\ell,4\ell,5\ell}) \) for \( \ell = 3 \) and 4 and zeros of \( P_{4,4,5}(e^{-s}) \)
Selberg zeta function

\[ Z(s) := \prod_{\gamma \in P_X} \prod_{m \geq 0} \left( 1 - e^{-(s+m)l(\gamma)} \right) \]
Proof: Selberg and dynamical zeta function

Selberg zeta function

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Z(s) has analytic continuation to \( \mathbb{C} \)

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\( Z(s) \) has analytic continuation to \( \mathbb{C} \)

\[ Z(s) = 0 \iff s \in \text{Res}(X) \text{ oder } s = 0, -1, -2, \ldots \]
(Patterson-Perry, 2001 Duke Math.)
Proof: Selberg and dynamical zeta function

Selberg zeta function

\[ Z(s) := \prod_{\gamma \in P_X} \prod_{m \geq 0} \left( 1 - e^{-(s+m)l(\gamma)} \right) \]

\( Z(s) \) has analytic continuation to \( \mathbb{C} \) (Guillopé, 1992 Adv.Stud.Pur.Math.)

\[ Z(s) = 0 \Leftrightarrow s \in \text{Res}(X) \text{ oder } s = 0, -1, -2, \ldots \]

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Construct a suitable dynamical system and transfer operator s.t.

\[ Z(s) = \det(1 - \mathcal{L}_s) \]
Proof: Geometric limit

Consider for the family $X_{n_1 \ell, n_2 \ell, n_3 \ell}$ the family of transfer operators $\mathcal{L}_{s; \ell}$. 
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Consider for the family $X_{n_1 \ell, n_2 \ell, n_3 \ell}$ the family of transfer operators $\mathcal{L}_{s;\ell}$.

Use Grothendieck expansion of Fredholm determinant

$$Z(s/\ell) = \det(1 - \mathcal{L}_{s/\ell}) = 1 + \sum_k d^{(k)}(s, \ell)$$
Proof: Geometric limit

Consider for the family $X_{n_1 \ell, n_2 \ell, n_3 \ell}$ the family of transfer operators $\mathcal{L}_{\ell, \ell}$.

Use Grothendieck expansion of Fredholm determinant

$$Z(s/\ell) = \det(1 - \mathcal{L}_{s/\ell}) = 1 + \sum_k d^{(k)}(s, \ell)$$

Use techniques of Jenkinson-Pollicott to get bounds on $d^{(k)}(s, \ell)$

$$\Rightarrow \sum_{k>6} d^{(k)}(s, \ell) \rightarrow 0$$
Proof: Geometric limit

Consider for the family $X_{n_1\ell,n_2\ell,n_3\ell}$ the family of transfer operators $\mathcal{L}_{s;\ell}$.

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$$\Rightarrow \sum_{k>6} d^{(k)}(s, \ell) \to 0$$

Use trace formula

$$\Rightarrow 1 + \sum_{k \leq 6} d^{(k)}(s, \ell) \to P_{n_1,n_2,n_3}(ze^{-s})$$
Resonance chains become “exact” in the limit $\ell \to \infty$. 
Summary and Outlook

- Resonance chains become “exact” in the limit $\ell \to \infty$.
- Approximate formula for individual resonance position
Resonance chains become “exact” in the limit $\ell \to \infty$.

Approximate formula for individual resonance position

Outlook:

higher-order formulas for chains
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Approximate formula for individual resonance position

Outlook:
- higher-order formulas for chains
- generalizations to other manifolds
The End
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THANK YOU! MERCI!