Resonance chains on Schottky surfaces

T. Weich[1]

[1]Universität Paderborn

2016 February 5, Grenoble

Resonances – a quantum dynamical motivation

H: Hamilton opertor of single particle open quantum system

Resonances – a quantum dynamical motivation

H: Hamilton opertor of single particle open quantum system

What are resonances?

Resonances – a quantum dynamical motivation

H: Hamilton opertor of single particle open quantum system

What are resonances?

Poles of the meromorphically continued resolvent $R(z) = \frac{1}{iH+z}$

Resonances – a quantum dynamical motivation

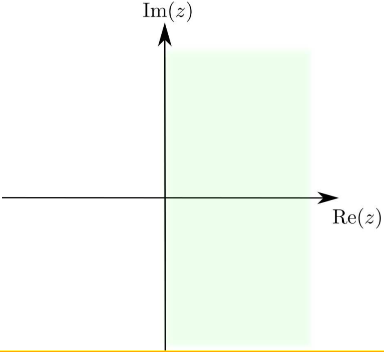
H: Hamilton opertor of single particle open quantum system

What are resonances?

Poles of the meromorphically continued resolvent $R(z) = \frac{1}{iH+z}$

Laplace transformation of time evolution operator:

$$\int_0^\infty \underbrace{e^{-iHt}}_{U(t)} e^{-zt} dt = \frac{e^{(-iH-z)t}}{-iH-z}\bigg|_0^\infty = \frac{1}{iH+z} = R(z)$$



Resonances – a quantum dynamical motivation

H: Hamilton opertor of single particle open quantum system

What are resonances?

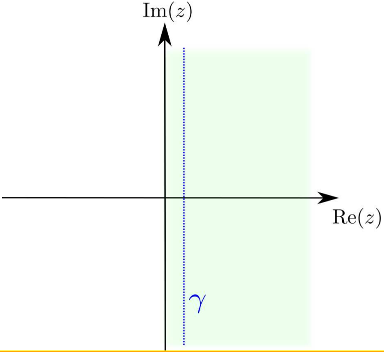
Poles of the meromorphically continued resolvent $R(z) = \frac{1}{iH+z}$

Laplace transformation of time evolution operator:

$$\int_0^\infty \underbrace{e^{-iHt}}_{U(t)} e^{-zt} dt = \left. \frac{e^{(-iH-z)t}}{-iH-z} \right|_0^\infty = \frac{1}{iH+z} = R(z)$$

Inversion formula:

$$U(t) = \frac{1}{2\pi i} \int_\gamma R(z) e^{zt} dz$$



Resonances – a quantum dynamical motivation

H: Hamilton opertor of single particle open quantum system

What are resonances?

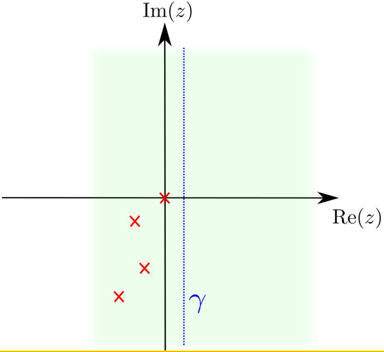
Poles of the meromorphically continued resolvent $R(z) = \frac{1}{iH+z}$

Laplace transformation of time evolution operator:

$$\int_0^\infty \underbrace{e^{-iHt}}_{U(t)} e^{-zt} dt = \frac{e^{(-iH-z)t}}{-iH-z}\Big|_0^\infty = \frac{1}{iH+z} = R(z)$$

Inversion formula:

$$U(t) = \frac{1}{2\pi i}\int_\gamma R(z) e^{zt} dz$$



Resonances – a quantum dynamical motivation

H: Hamilton opertor of single particle open quantum system

What are resonances?

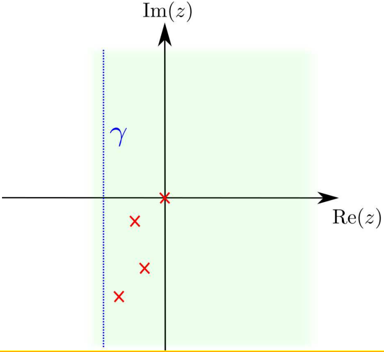
Poles of the meromorphically continued resolvent $R(z) = \frac{1}{iH+z}$

Laplace transformation of time evolution operator:

$$\int_0^\infty \underbrace{e^{-iHt}}_{U(t)} e^{-zt} dt = \frac{e^{(-iH-z)t}}{-iH-z}\Bigg|_0^\infty = \frac{1}{iH+z} = R(z)$$

Inversion formula:

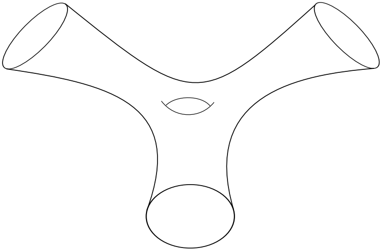
$$U(t) = \frac{1}{2\pi i}\int_\gamma R(z) e^{zt} dz = \sum_{\lambda_i} e^{\lambda_i t}\Pi_\lambda + \mathcal{O}(e^{-Ct})$$



Schottky surfaces

Def. Schottky surface:

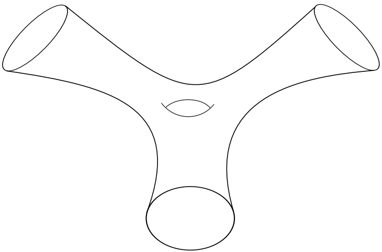
- non-compact surface of constant negative curvature ($X = \Gamma\backslash\mathbb{H}^2$)
- finite genus
- finite number of funnels
- no cusps



Schottky surfaces

Def. Schottky surface:

- non-compact surface of constant negative curvature ($X = \Gamma\backslash\mathbb{H}^2$)
- finite genus
- finite number of funnels
- no cusps

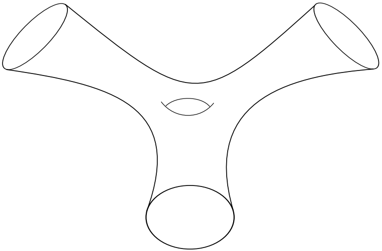


Hamiltonian: $H = -\Delta_X \leftrightarrow$ "free quantum particle"

Schottky surfaces

Def. Schottky surface:

- non-compact surface of constant negative curvature ($X = \Gamma\backslash\mathbb{H}^2$)
- finite genus
- finite number of funnels
- no cusps



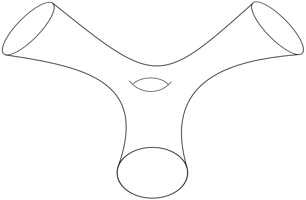
Hamiltonian: $H = -\Delta_X \leftrightarrow$ "free quantum particle"

Neg. curvature $\Rightarrow$ geodesic flow is chaotic

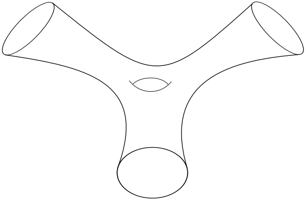
$\Rightarrow$ Model for open quantum chaotic system

(Relations to number theory)

Resonances



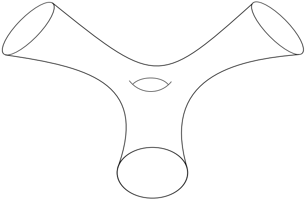
Resonances

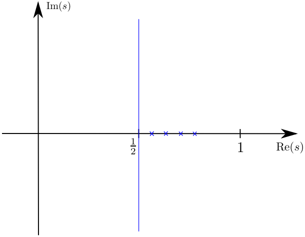


X has infinite volume $\Rightarrow$

$$\begin{aligned} \sigma^{\text{disc}}_{L^2}(\Delta_X) &\subset (0, 1/4) \text{ finite} \\ \sigma^{\text{cont}}_{L^2}(\Delta_X) &= [1/4, \infty) \end{aligned}$$

Resonances





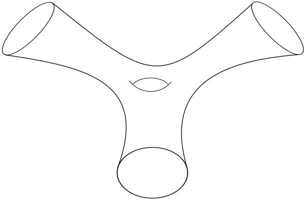
X has infinite volume $\Rightarrow$

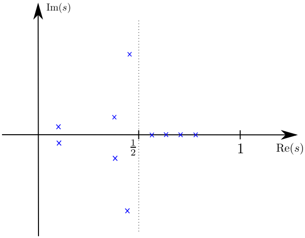
$$\sigma^{\text{disc}}_{L^2}(\Delta_X) \subset (0, 1/4) \text{ finite}$$
$$\sigma^{\text{cont}}_{L^2}(\Delta_X) = [1/4, \infty)$$

resolvent:

$$R_X(s) := (\Delta_X - s(1-s))^{-1} : L^2 \to L^2$$

Resonances





X has infinite volume $\Rightarrow$

$$\begin{aligned} \sigma^{\mathsf{disc}}_{L^2}(\Delta_X) &\subset (0,1/4) \text{ finite} \\ \sigma^{\mathsf{cont}}_{L^2}(\Delta_X) &= [1/4,\infty) \end{aligned}$$

resolvent:

$$R_X(s) := (\Delta_X - s(1-s))^{-1} : L^2_{\mathsf{comp}} \to L^2_{\mathsf{loc}}$$

$R_X(s)$ is meromorphic family of operators with poles of finite multiplicity (Mazzeo-Melrose 86)

$$\mathsf{Res}(X) := \{\text{poles of } R_X(s)\}$$

Fractal Weyl law

Spectral Geometry:

$$\text{Spectral properties} \quad \leftrightsquigarrow \quad \text{Geometry}$$

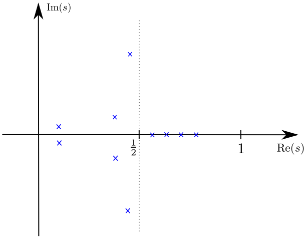
Fractal Weyl law

Spectral Geometry:

$$\text{Spectral properties} \quad \leftrightsquigarrow \quad \text{Geometry}$$

Let $C_0 \in \mathbb{R}$

$$N_{C_0}(k) := \#\{s \in \mathsf{Res}(X), \mathsf{Re}(s) > C_0, |\mathsf{Im}(s)| < k\}$$



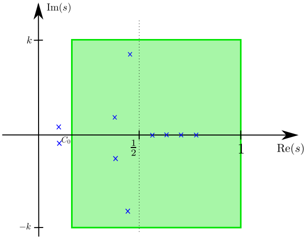
Fractal Weyl law

Spectral Geometry:

Spectral properties $\leftrightsquigarrow$ Geometry

Let $C_0 \in \mathbb{R}$

$$N_{C_0}(k) := \#\{s \in \mathsf{Res}(X), \mathsf{Re}(s) > C_0, |\mathsf{Im}(s)| < k\}$$



Fractal Weyl law

Spectral Geometry:

$$\text{Spectral properties} \quad \leftrightsquigarrow \quad \text{Geometry}$$

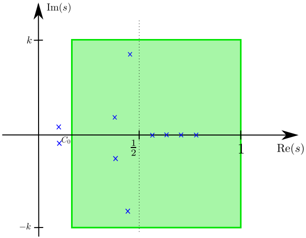
Let $C_0 \in \mathbb{R}$

$$N_{C_0}(k) := \#\{s \in \mathrm{Res}(X), \mathrm{Re}(s) > C_0, |\mathrm{Im}(s)| < k\}$$

then there is $C > 0$ with

$$N_{C_0}(k) \leq Ck^{1+\delta_\Gamma}$$

(Zworski, 1999 Invent.Math.)



Fractal Weyl law

Spectral Geometry:

$$\text{Spectral properties} \quad \rightsquigarrow \quad \text{Geometry}$$

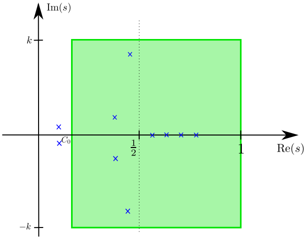
Let $C_0 \in \mathbb{R}$

$$N_{C_0}(k) := \#\{s \in \mathsf{Res}(X), \mathsf{Re}(s) > C_0, |\mathsf{Im}(s)| < k\}$$

then there is $C > 0$ with

$$N_{C_0}(k) \leq C k^{1+\delta_\Gamma}$$

(Zworski, 1999 Invent.Math.)
Remark: $\delta_\Gamma = \dim_H(\Lambda_\Gamma)$



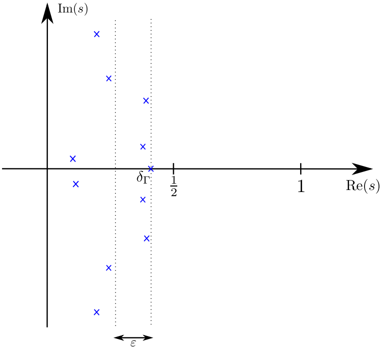
Conjecture: Exponent is sharp

Spectral Gap

Theorem (Naud 2005)

There is $\varepsilon > 0$, such that

$$\#\{s \in \mathsf{Res}(X), \mathsf{Re}(s) > \delta_\Gamma - \varepsilon\} < \infty$$

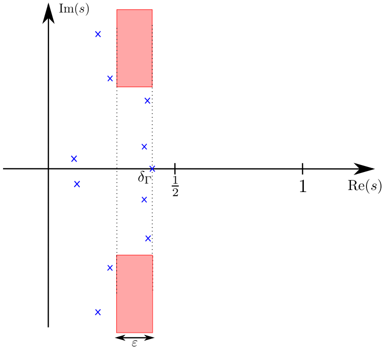


Spectral Gap

Theorem (Naud 2005)

There is $\varepsilon > 0$, such that

$$\#\{s \in \mathsf{Res}(X), \mathsf{Re}(s) > \delta_\Gamma - \varepsilon\} < \infty$$

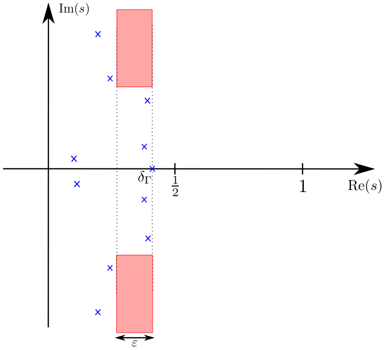


Spectral Gap

Theorem (Naud 2005)

There is $\varepsilon > 0$*, such that*

$$\#\{s \in \mathsf{Res}(X), \mathsf{Re}(s) > \delta_\Gamma - \varepsilon\} < \infty$$



Conjecture: $\varepsilon = \delta_\Gamma/2$

Numerical Tests of Conjectures

Borthwick 2014 Exp.Math.

- Fractal Weyl Conjecture (✓)

Numerical Tests of Conjectures

Borthwick 2014 Exp.Math.

- Fractal Weyl Conjecture (✓)
- Gap Conjecture (✗)

Numerical Tests of Conjectures

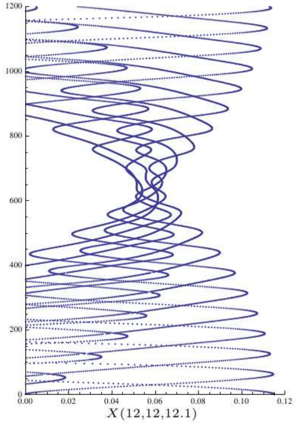
Borthwick 2014 Exp.Math.

- Fractal Weyl Conjecture (✓)
- Gap Conjecture (✗)
- Observation of resonance chains:

Numerical Tests of Conjectures

Borthwick 2014 Exp.Math.

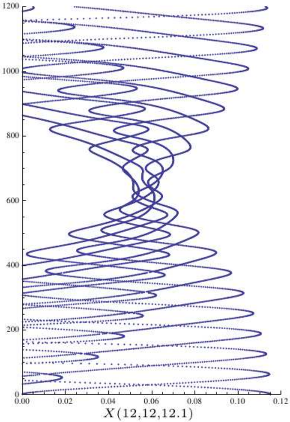
- Fractal Weyl Conjecture (✓)
- Gap Conjecture (✗)
- Observation of resonance chains:

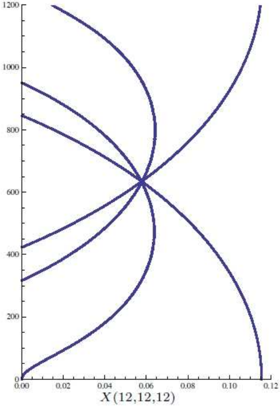


Numerical Tests of Conjectures

Borthwick 2014 Exp.Math.

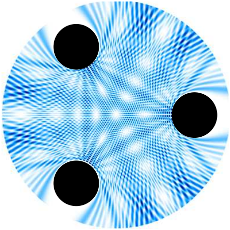
- Fractal Weyl Conjecture (✓)
- Gap Conjecture (✗)
- Observation of resonance chains:

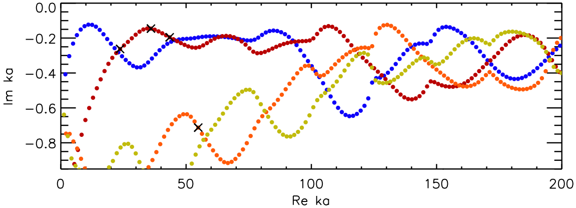




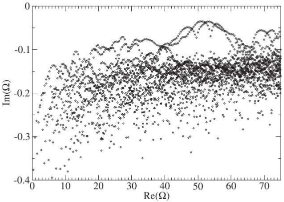
Resonance chains in physics

3-disk system:

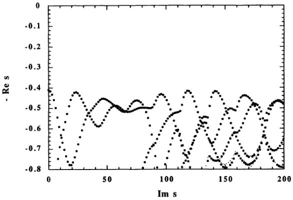




microdisk cavity/laser
(Main Wiersig 2008):



Ruelle-Resonances
(Gaspard-Ramires 1992):



Resonance Chains

Questions

1) On which surfaces do we see chains?

Resonance Chains

Questions

1) On which surfaces do we see chains?
2) Approximative formula for these chains?

Resonance Chains

Questions

1) On which surfaces do we see chains?
2) Approximative formula for these chains?

Concerning 1):

S. Barkhofen, F. Faure, T.W., 2014

Resonance chains $\leftrightarrow$ Clustering of length spectrum

Resonance Chains

Questions

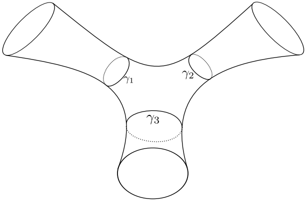
1) On which surfaces do we see chains?
2) Approximative formula for these chains?

Concerning 1):

S. Barkhofen, F. Faure, T.W., 2014

Resonance chains $\leftrightarrow$ Clustering of length spectrum

Concerning 2):



X_{l_1,l_2,l_3}

Resonance Chains

Questions

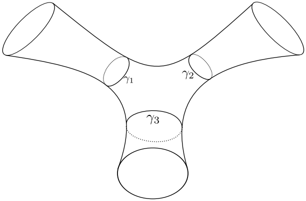
1) On which surfaces do we see chains?
2) Approximative formula for these chains?

Concerning 1):

S. Barkhofen, F. Faure, T.W., 2014

Resonance chains $\leftrightarrow$ Clustering of length spectrum

Concerning 2):

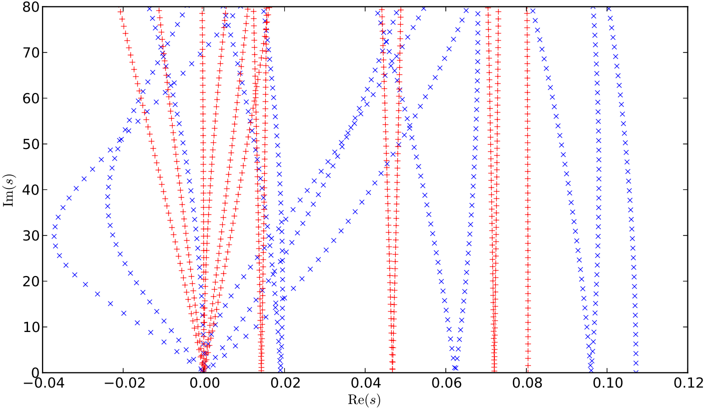


X_{l_1,l_2,l_3}

$\Rightarrow$ study $X_{n_1\ell,n_2\ell,n_3\ell}$
in the limit $\ell \to \infty$

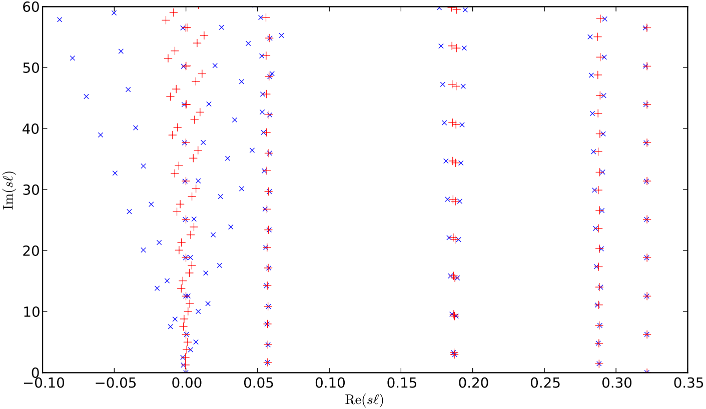
Rescaled resonances

$\mathrm{Res}(X_{4\ell,4\ell,5\ell})$ for $\ell = 3$ and 4



Rescaled resonances

$\ell \cdot \text{Res}(X_{4\ell,4\ell,5\ell})$ for $\ell = 3$ and 4



Existence of resonance chains

Theorem (T.W., 2015)

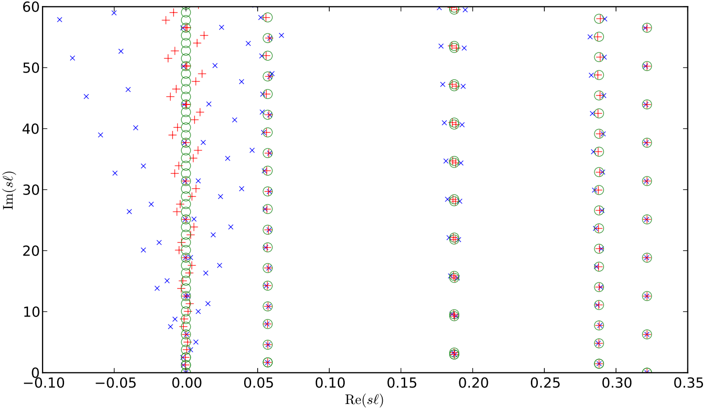
Suppose $n_i + n_j > n_k$ *and define*

$$\begin{aligned} P_{n_1,n_2,n_3}(x) \quad := \quad & 1 - 2(x^{n_1} + x^{n_2} + x^{n_3}) + x^{2n_1} + x^{2n_2} + x^{2n_3} \\ & +2(x^{n_1+n_2} + x^{n_2+n_3} + x^{n_1+n_3}) - 4x^{n_1+n_2+n_3}. \end{aligned}$$

Then "an arbitrary large fixed number of rescaled resonances converges in the limit $\ell \to \infty$ *uniformly to the zeros of* $P_{n_1,n_2,n_3}(e^{-s})$*".*

Numerical Test

$\ell \cdot \mathrm{Res}(X_{4\ell,4\ell,5\ell})$ for $\ell = 3$ and 4 and zeros of $P_{4,4,5}(e^{-s})$



Proof: Selberg and dynamical zeta function

- Selberg zeta function

$$Z(s) := \prod_{\gamma \in P_X} \prod_{m \geq 0} \left(1 - e^{-(s+m)l(\gamma)}\right)$$

Proof: Selberg and dynamical zeta function

- Selberg zeta function

$$Z(s) := \prod_{\gamma \in P_X} \prod_{m \geq 0} \left(1 - e^{-(s+m)l(\gamma)}\right)$$

- $Z(s)$ has analytic continuation to $\mathbb{C}$
(Guillopé, 1992 Adv.Stud.Pur.Math.)

Proof: Selberg and dynamical zeta function

- Selberg zeta function

$$Z(s) := \prod_{\gamma \in P_X} \prod_{m \geq 0} \left(1 - e^{-(s+m)l(\gamma)}\right)$$

- $Z(s)$ has analytic continuation to $\mathbb{C}$
(Guillopé, 1992 Adv.Stud.Pur.Math.)

- $$Z(s) = 0 \Leftrightarrow s \in \mathrm{Res}(X) \text{ oder } s = 0, -1, -2, \ldots$$

  (Patterson-Perry, 2001 Duke Math.)

Proof: Selberg and dynamical zeta function

- Selberg zeta function

$$Z(s) := \prod_{\gamma \in P_X} \prod_{m \geq 0} \left(1 - e^{-(s+m)l(\gamma)}\right)$$

- $Z(s)$ has analytic continuation to $\mathbb{C}$
(Guillopé, 1992 Adv.Stud.Pur.Math.)

- $$Z(s) = 0 \Leftrightarrow s \in \mathrm{Res}(X) \text{ oder } s = 0, -1, -2, \ldots$$

(Patterson-Perry, 2001 Duke Math.)

- Construct a *suitable* dynamical system and transfer operator s.t.

$$Z(s) = \det(1 - \mathcal{L}_s)$$

Proof: Geometric limit

- Consider for the family $X_{n_1\ell,n_2\ell,n_3\ell}$ the family of transfer operators $\mathcal{L}_{s;\ell}$.

Proof: Geometric limit

- Consider for the family $X_{n_1\ell,n_2\ell,n_3\ell}$ the family of transfer operators $\mathcal{L}_{s;\ell}$.
- Use Grothendieck expansion of Fredholm determinant

$$Z(s/\ell) = \det(1 - \mathcal{L}_{s/\ell}) = 1 + \sum_k d^{(k)}(s,\ell)$$

Proof: Geometric limit

- Consider for the family $X_{n_1\ell,n_2\ell,n_3\ell}$ the family of transfer operators $\mathcal{L}_{s;\ell}$.
- Use Grothendieck expansion of Fredholm determinant

$$Z(s/\ell) = \det(1 - \mathcal{L}_{s/\ell}) = 1 + \sum_k d^{(k)}(s, \ell)$$

- Use techniques of Jenkinson-Pollicott to get bounds on $d^{(k)}(s, \ell)$

$$\Rightarrow \sum_{k>6} d^{(k)}(s, \ell) \to 0$$

Proof: Geometric limit

- Consider for the family $X_{n_1\ell,n_2\ell,n_3\ell}$ the family of transfer operators $\mathcal{L}_{s;\ell}$.
- Use Grothendieck expansion of Fredholm determinant

$$Z(s/\ell) = \det(1 - \mathcal{L}_{s/\ell}) = 1 + \sum_k d^{(k)}(s,\ell)$$

- Use techniques of Jenkinson-Pollicott to get bounds on $d^{(k)}(s,\ell)$

$$\Rightarrow \sum_{k>6} d^{(k)}(s,\ell) \to 0$$

- Use trace formula

$$\Rightarrow 1 + \sum_{k\leq 6} d^{(k)}(s,\ell) \to P_{n_1,n_2,n_3}(ze^{-s})$$

Summary and Outlook

- Resonance chains become "exact" in the limit $\ell \to \infty$.

Summary and Outlook

- Resonance chains become "exact" in the limit $\ell \to \infty$.
- Approximate formula for individual resonance position

Summary and Outlook

- Resonance chains become "exact" in the limit $\ell \to \infty$.
- Approximate formula for individual resonance position

Outlook:

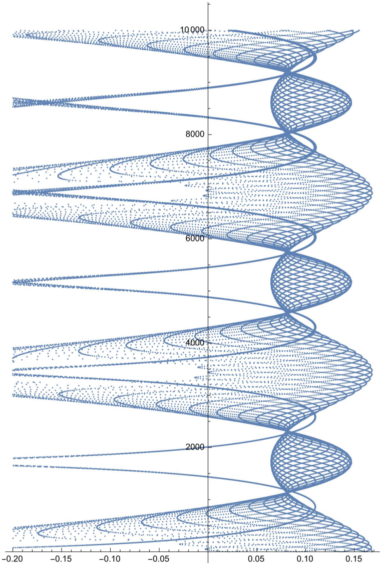
- higher-order formulas for chains

Summary and Outlook

- Resonance chains become "exact" in the limit $\ell \to \infty$.
- Approximate formula for individual resonance position

Outlook:

- higher-order formulas for chains
- generalizations to other manifolds



The End

The End

THANK YOU! MERCI!