Spectra of large non-self-adjoint Toeplitz matrices subject to small random perturbations

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Outline

Non-self-adjoint operators and spectral instability

Random perturbations of NSA Toeplitz matrices

Large Jordan block matrices

Large bi-diagonal matrices - first results

Outlook
Non-self-adjoint operators and spectral instability

Non-self-adjoint operators appear naturally in many areas, e.g.:

- In the theory of linear PDEs given by non-normal operators
  - solvability theory
  - evolution equations given by non-normal operators
  - the Kramers-Fokker-Planck type operators
  - the damped wave equation
  - ...

- In mathematical physics when studying scattering poles (resonances).

**Bad resolvent control:** For non-normal operators $P : \mathcal{H} \rightarrow \mathcal{H}$ the norm of the resolvent may be very large even far away from the spectrum $\sigma(P)$:

$$\| (P - z)^{-1} \| \gg \frac{1}{\text{dist}(z, \sigma(P))}.$$  

**Consequence:**

- The spectrum can be very unstable under small perturbations of the operator.
Pseudospectrum

A way to quantify this zone of spectral instability is given by the $\varepsilon$-pseudospectrum [Trefethen-Embree ’05], defined by

$$\sigma_\varepsilon(P) := \sigma(P) \cup \{z \in \rho(P) : \| (P - z)^{-1} \| > \varepsilon^{-1} \};$$

or equivalently

$$\sigma_\varepsilon(P) := \bigcup_{Q \in \mathcal{B}(\mathcal{H}) \atop \|Q\| < \varepsilon} \sigma(P + Q).$$

▶ Renewed interest has started in numerical analysis with the works of [Trefethen ’97] (and [Trefethen-Embree ’05]);

▶ Active subject in the field of PDE: Davies, Zworski, Sjöstrand, Bulton, Pravda-Starov, ... ;

▶ It is natural to add small random perturbations; Hager ’06, Hager-Sjöstrand ’08, Bordeaux-Montrieux ’08, Davies-Hager ’09, Sjöstrand ’08–’15, Zworski-Christiansen ’10.
Laurent and Toeplitz operators

Laurent operator: For $p \in L^\infty(S^1)$, $L(p) : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$L(p)u = \hat{p} * u, \quad (L(p)u)(n) = \frac{1}{2\pi} \int_{S^1} p(\xi)\hat{u}(\xi)e^{in\xi} d\xi.$$ 

Toeplitz operator: For $p \in L^\infty(S^1)$, $T(p) : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$T(p) := 1_{\mathbb{N}} L(p) 1_{\mathbb{N}}.$$ 

Identifying $\mathbb{C}^N$ with $\ell^2([1, N])$, we define a $N \times N$ Toeplitz matrix by

$$T_N(p) = 1_{[1,N]} L(p) 1_{[1,N]},$$

or by its matrix representation

$$T_N(p) = \begin{pmatrix}
    a_0 & a_{-1} & a_{-2} & \ldots & a_{1-N} \\
    a_1 & a_0 & a_{-1} & \ddots & \vdots \\
    \vdots & a_1 & a_0 & a_{-1} & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    a_{N-1} & \ldots & \ldots & a_0 & a_1 \\
\end{pmatrix}, \quad a_j = \hat{a}(j) \in \mathbb{C}$$
Spectra and Pseudospectra

The spectral theory of these operators were extensively studied, cf. Böttcher-Silbermann (also Trefethen-Embree), Widom, Goldsheid-Khoruzenko, Hatano-Nelson, Ghoberg, ...

We assume for simplicity that the symbol $p$ is a trigonometric polynomial (i.e. the corresponding operators are banded).

i) **Laurent operator**: is normal, so $\sigma(L(p)) = p(S^1)$.

ii) **Toeplitz operator**: by truncating we may lose normality, and by [Gohberg '52]

$$\sigma_{ess}(T(p)) = p(S^1) \text{ and } \sigma(T(p)) = p(S^1) \cup \{ z \in \mathbb{C}; \ wind(p, z) \neq 0 \}$$

iii) **Toeplitz matrix**:

- for non-normal $T_N(p)$, in general $\lim_{N \to \infty} \sigma(T_N(p)) \neq \lim_{N \to \infty} \sigma(T(p))$.

- Set $p_r(z) = p(rz)$, then by [Schmidt-Spitzer '60]

$$\lim_{N \to \infty} \sigma(T_N(p)) = \bigcap_{r > 0} \sigma(T(p_r)),$$

$$\sigma(T_N(p)) \subset \text{a finite union of analytic connected arcs}.$$ 

Pseudospectra are well behaved!

i) $\lim_{N \to \infty} \sigma_\varepsilon(T_N(p)) = \sigma_\varepsilon(T(p))$  [Landau '75, Reichel-Trefethen '92, Böttcher '96]

ii) For every $\varepsilon > 0$, there exists $N_0$ s.t. for all $N \geq N_0$, $\sigma(T(p)) \subset \sigma_\varepsilon(T_N(p))$
Example: \( p(z) = 2z^{-3} - z^{-2} + 2iz^{-1} - 4z^2 - 2iz^3 \)

**Figure:** Picture taken from [Trefethen-Embree '05], represents the spectra of the Laurent, Toeplitz operators and Toeplitz and circulant matrix corresponding to the symbol \( p \).
Small random perturbations of large Toeplitz matrices

\[ p(z) = 2z^{-3} - z^{-2} + 2iz^{-1} - 4z^2 - 2iz^3 \]
\[ p(z) = -z^{-4} - (3 + 2i)z^{-3} + iz^{-2} + z^{-1} + 10z + (3 + i)z^2 + 4z^3 + iz^4 \]
\[ p(z) = 2iz^{-1} + z^2 + \frac{7}{10}z^3 \]
Small random perturbations of large Toeplitz matrices

We are interested in small random perturbations of $P^0_N : \mathbb{C}^N \to \mathbb{C}^N$, a non-normal Toeplitz matrix for $N \gg 1$, of the form:

$$P^\delta,\omega := P^0_N + \delta Q_\omega, \quad 0 < \delta \ll 1$$

where

$$Q_\omega = (q_{j,k}(\omega))_{1 \leq j, k \leq N} \quad \text{with} \quad q_{j,k} \sim \mathcal{N}_\mathbb{C}(0, 1) \quad \text{(iid).}$$

- If $C_1 > 0$ is large enough, then

$$q \in B_{\mathbb{C}^{N^2}}(0, C_1 N) \iff \|Q\|_{\text{HS}} \leq C_1 N, \quad \text{with probability} \quad \geq 1 - e^{-N^2}.$$

We study the following two cases of $P^0_N$:

- **Case 1:** Large Jordan block matrices
- **Case 2:** Large bi-diagonal matrices
Perturbations of large Jordan blocks

We are interested in the spectrum of a random perturbation of the large Jordan block $A_0$:

$$A_0 = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & 0 & \ldots & 0
\end{pmatrix} : \mathbb{C}^N \to \mathbb{C}^N.$$

- The spectrum of $A_0$ is $\sigma(A_0) = \{0\}$;
- $D(0,1)$ is a region of spectral instability;
- in $\mathbb{C} \setminus D(0,1)$ we have spectral stability, i.e. a good resolvent estimate.

For a small ($0 < \delta \ll 1$) (random) perturbation

$$A_\delta = A_0 + \delta Q_\omega, \quad Q_\omega = (q_{j,k}(\omega))_{1 \leq j, k \leq N}, \quad q_{j,k}(\omega) \sim \mathcal{N}_\mathbb{C}(0, 1) \text{ (iid)}$$

we expect the spectrum to move in a small neighborhood of $\overline{D(0,1)}$. 
Numerical simulation for the eigenvalues of $A_\delta$, a complex Gaussian random perturbation of $A_0$, with $p(z) = z$. 

$D = 601 \quad \delta = 1e-12$
Previous results

**Theorem (Davies-Hager ’09)**

If $0 < \delta \leq N^{-7}$, $R = \delta^{1/N}$, $\sigma > 0$, then with probability $\geq 1 - 2N^{-2}$, we have $\sigma(A_\delta) \subset D(0, RN^{3/N})$ and

$$
\#(\sigma(A_\delta) \cap D(0, Re^{-\sigma})) \leq \frac{2}{\sigma} + \frac{4}{\sigma} \ln N.
$$

- With probability close to 1, **most eigenvalues** are close to a circle, contained in $D(0, RN^{3/N}) \setminus D(0, Re^{-\sigma})$.
- At most $O(\ln N)$ eigenvalues inside $D(0, Re^{-\sigma})$.
- Sjöstrand improved on this result by giving a probabilistic angular Weyl law for the eigenvalues close to the $S^1$

**Theorem (Guionnet-Matched-Wood-Zeitouni ’14)**

Assume that $N^{-1-\kappa'} \leq \delta \leq N^{-1-\kappa}$ for some $0 < \kappa < \kappa'$, then

$$
\frac{1}{N} \sum_{\mu \in \sigma(A_\delta)} \delta(z - \mu) \to \text{the uniform measure on } S^1,
$$

weakly in probability as $N \to \infty$. 
**Interior density of eigenvalues**

To obtain more information in the interior, we consider the random point process (related works on the zeros of random polynomials by Shiffman and Zelditch, Sodin, Hough-Krishnapur-Peres-Virág)

\[
\Xi = \sum_{z \in \sigma(A_\delta)} \delta_z,
\]

Study for \( \varphi \in C_0(D(0, r)) \) the first intensity measure of \( \Xi \):

\[
E[\Xi(\varphi)1_{B(0, C_1N)}] = \int \varphi(z) d\nu(z) \quad \text{(recall: } \|Q\|_{HS} \leq C_1N)\).
\]

**Theorem (Sjöstrand-V ’14)**

Let \( e^{-N/C} \leq \delta \ll N^{-3} \) and \( N \gg 1 \). Let \( r_0 \) belong to a parameter range,

\[
\frac{1}{C} \leq r_0 \leq 1 - \frac{1}{N}, \quad \text{s.t.} \quad \frac{r_0^{N-1}N}{\delta} (1 - r_0)^2 + \delta N^3 \ll 1.
\]

Then, for all \( \varphi \in C_0(D(0, r_0 - 1/N)) \)

\[
E \left[ 1_{B_{C_1N^2}(0, C_1N)}(q) \sum_{\lambda \in \sigma(A_\delta)} \varphi(\lambda) \right] = \frac{1}{2\pi} \int \varphi(z)\xi(z)L(dz),
\]

where

\[
\xi(z) = \frac{4}{(1 - |z|^2)^2} \left( 1 + O\left( \frac{|z|^{N-1}N}{\delta} (1 - |z|)^2 + \delta N^3 \right) \right).
\]
Figure: The experimental integrated density of eigenvalues (averaged over 500 realizations), as a function of the radius, of a $1001 \times 1001$-Jordan block matrix perturbed with a random complex Gaussian matrix and with coupling $\delta = 2 \cdot 10^{-10}$. The red line is the hyperbolic volume on the unit disk as a function of the radius.
Large bi-diagonal matrices - first results

We now consider the following two cases:

\[
P_I = \begin{pmatrix} 0 & a & 0 & \cdots & 0 \\ b & 0 & a & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & b & 0 \end{pmatrix} \quad \text{and} \quad P_{II} = \begin{pmatrix} 0 & a & b & 0 & \cdots & \cdots & 0 \\ 0 & 0 & a & b & \cdots & \cdots & 0 \\ 0 & 0 & 0 & a & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a & b \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a \end{pmatrix}.
\]

▶ Here \( a, b \in \mathbb{C} \setminus \{0\} \) and \( N \gg 1 \).

▶ Identifying \( \mathbb{C}^N \) with \( \ell^2([1, N]) \), \( [1, N] = \{1, 2, \ldots, N\} \) and also with \( \ell^2_{[1, N]}(\mathbb{Z}) \) (the space of all \( u \in \ell^2(\mathbb{Z}) \) with support in \( [1, N] \)), we have:

\[
P_I = 1_{[1, N]}(ae^{iDx} + be^{-iDx}),
\]

\[
P_{II} = 1_{[1, N]}(ae^{iDx} + be^{2iDx}).
\]

▶ The symbols of these operators are,

\[
p_I(\xi) = ae^{i\xi} + be^{-i\xi}, \quad p_{II}(\xi) = ae^{i\xi} + be^{2i\xi}.
\]
Numerical simulation for $P_{ll}$

$D = \frac{601}{\delta} = 1e-12$

$\sigma(P_{\delta})$

$\sigma(P_0)$

$p(S^1)$
Numerical simulation for $P_I$
Theorem (Sjöstrand-V ’15)

Let \( P = P_1 \) where \( a, b \in \mathbb{C} \) satisfy \( 0 < |b| < |a| \). Let \( P_\delta = P + \delta Q_\omega \). Choose \( \delta \propto N^{-\kappa}, \kappa > 5/2 \) and consider the limit of large \( N \). Let \( \gamma \) be a segment of the ellipse \( E_1 = P_1(S^1) \) and let \( \Gamma = \Gamma(r, \gamma) = \{ z \in \mathbb{C}; \text{dist}(z, E_1) = \text{dist}(z, \gamma) < r \} \) with \((\ln N)/N \ll r \ll 1\). Let \( \delta_0 \) be small and fixed.

Then with probability

\[
\geq 1 - O(1) \left( \frac{1}{r} + \ln N \right) N^{2\kappa} e^{-2N\delta_0},
\]

we have

\[
\left| \#(\sigma(P_\delta) \cap \Gamma) - \frac{1}{2\pi} \text{vol}_{[0,N] \times S^1} P_1^{-1}(\Gamma) \right| \leq O(1) N^{\delta_0} \left( \frac{1}{r} + \ln N \right).
\]

\( \triangleright \) If we choose \( \gamma = E^1 \), we have

\[
\frac{1}{2\pi} \text{vol}_{[0,N] \times S^1} P_1^{-1}(\Gamma) = N
\]

(= total number of eigenvalues of \( P_\delta \)), so the number of eigenvalues outside of \( \Gamma \) is bounded be the right hand side of (2).

\( \triangleright \) With \( r > 0 \) fixed but arbitrarily small we get

Corollary

Let \( \Gamma \) be any fixed neighborhood of \( E_1 \). Then with probability as in (1), we have

\[
\left| \# (\sigma(P_\delta) \cap (\mathbb{C} \setminus \Gamma)) \right| \leq O(1) N^{\delta_0} \ln N.
\]
Outlook

1. Consider general non-normal Toeplitz matrices.
2. Density in the interior of the pseudospectrum
3. Correlation functions, universality
4. Limiting point-process
Thank you for your attention!

\[ D = 601 \quad \delta = 1e-12 \]