Energy statistics of quantum statistical systems in the adiabatic limit and Landauer's principle

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Tristan Benoist
joint work with M. Fraas, V. Jakšić and C.-A. Pillet

CNRS - LPT Toulouse, Université Paul Sabatier

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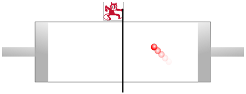
Szilard's engine

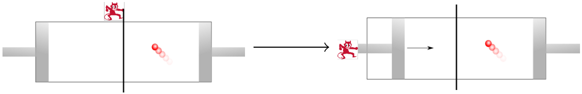
Particle at inverse temperature β.



Particle position recording

Szilard's engine

Particle at inverse temperature β.



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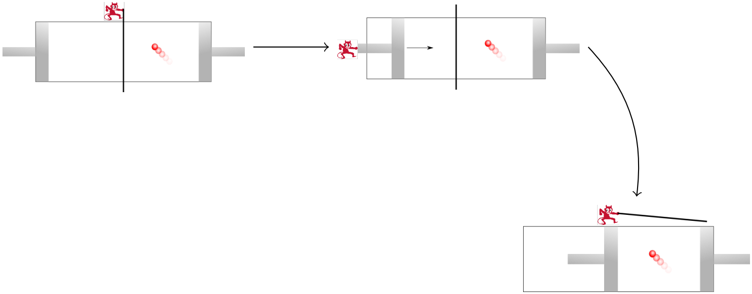

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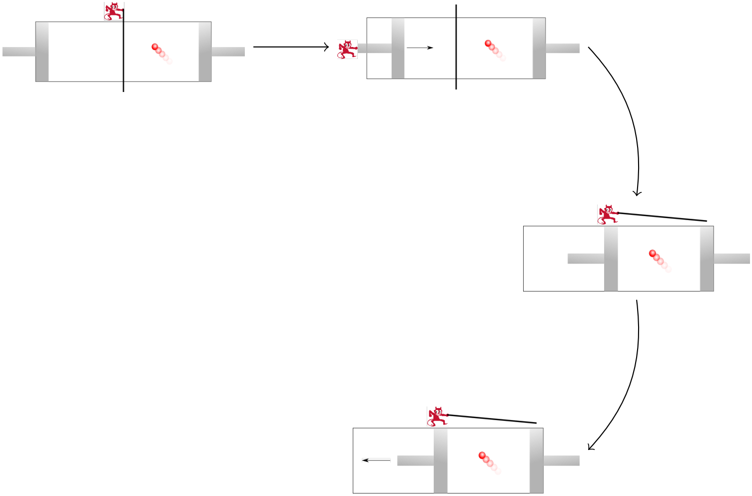

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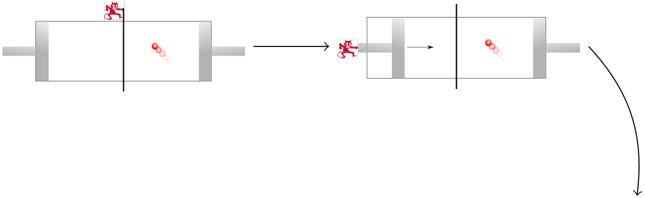
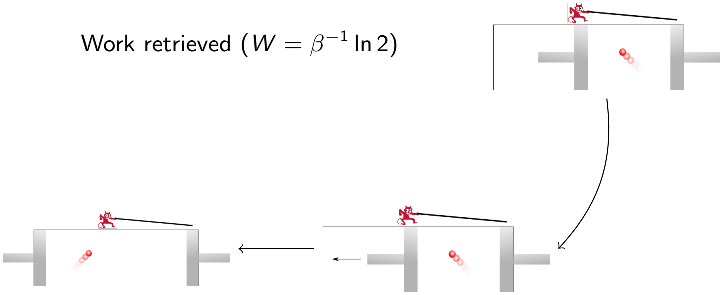


Work retrieved ($W = \beta^{-1} \ln 2$)



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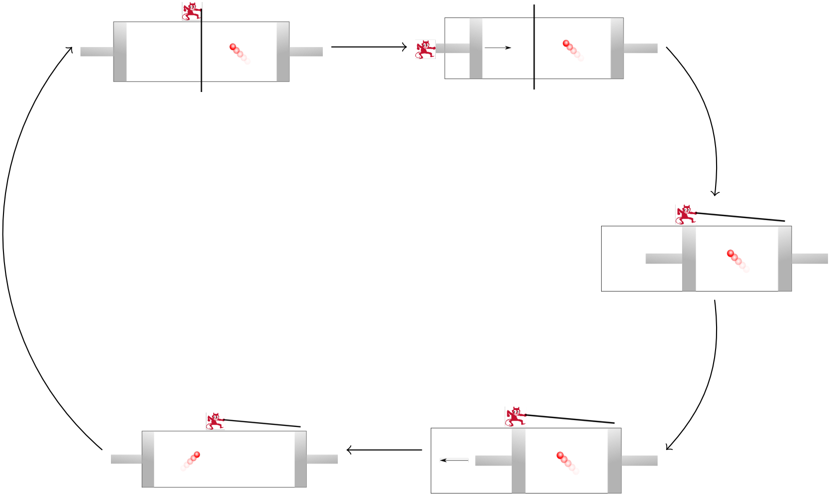

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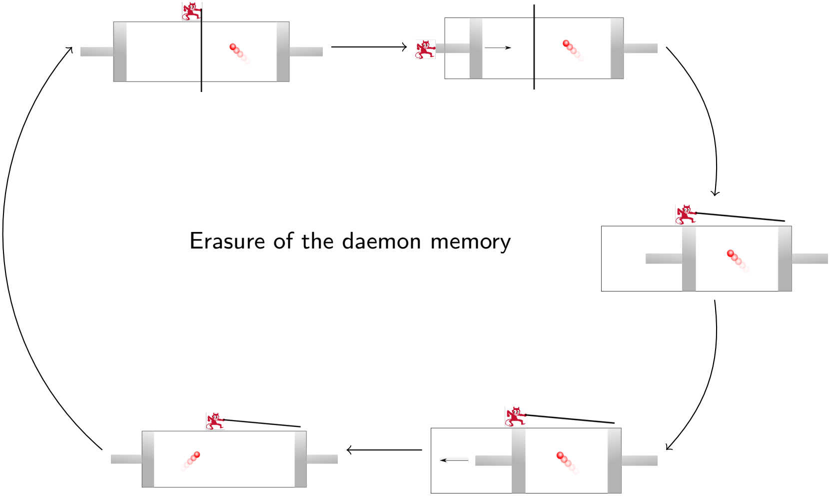


Landauer's principle

Information stored on a physical device [Landauer '61].
Heat cost of a bit of information erasure:

$$\langle \Delta Q \rangle \geq \beta^{-1} \ln 2.$$

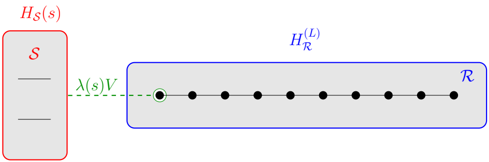
Landauer's principle is equivalent to the second law of thermodynamics [Reeb, Wolf '14; Jakšić, Pillet '14] (the bit is encoded on a qbit)

$$\Delta S + \langle \sigma \rangle = \beta \langle \Delta Q \rangle.$$

Quasi–static process and second law saturation

In the adiabatic(quasi–static) limit do we recover the saturation of the second law and Landauer's bound ?

A qbit erasure model



- System (memory) $\mathcal{S}$:
  $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2, \quad \mathcal{O}_{\mathcal{S}} = \mathcal{B}(\mathbb{C}^2), \quad \rho_{\mathcal{S}} \in \mathcal{B}(\mathbb{C}^2), \quad \rho_{\mathcal{S}} \geq 0, \mathrm{tr}\rho_{\mathcal{S}} = 1.$
- Heat bath $\mathcal{R}$: $\mathcal{H}_{\mathcal{R}}^{(L)} = \Gamma_a(\ell^2(\{1,\dots,L\})), \quad \mathcal{O}_{\mathcal{R}} = \mathrm{CAR}(\ell^2(\mathbb{N})).$

$$\rho_{\mathcal{R}}^{(L)} = e^{-\beta H_{\mathcal{R}}^{(L)}}/\mathrm{tr}(e^{-\beta H_{\mathcal{R}}^{(L)}}) \quad \xrightarrow[L\to\infty]{} \quad \rho_{\mathcal{R}} : \mathcal{O}_{\mathcal{R}} \to \mathbb{C}.$$

- Joint system $\mathcal{S} + \mathcal{R}$: $\mathcal{O} = \mathcal{O}_{\mathcal{S}} \otimes O_{\mathcal{R}}, \quad \rho : \mathcal{O} \to \mathbb{C}, \quad \rho \geq 0, \rho(I) = 1.$

Time dependent evolution

- Hamiltonian:

$$H^{(L)} : [0,1] \ni s \mapsto H_{\mathcal{S}}(s) + H_{\mathcal{R}}^{(L)} + \lambda(s)V.$$

with $H_{\mathcal{S}} : [0,1] \ni s \mapsto \mathcal{O}_{\mathcal{S},s.a.}$ and $V \in \mathcal{O}_{s.a.}$.
$(H_{\mathcal{R}}^{(L)} = d\Gamma(\kappa\Delta^{(L)}),\ H_{\mathcal{S}}(s) = \epsilon(s)I_2 + \delta(s)\sigma_z,\ V = \sigma_+ \otimes c(\delta_1) + \sigma_- \otimes c^*(\delta_1).)$

- Two time scales: $s \in [0,1]$, the epoch and $t = sT \in [0,T]$ the physical time.
- Time dependent evolution $\forall A \in \mathcal{O}$:

$$\partial_s \tau_T^s(A) = T\tau_T^s(\delta_{\mathcal{R}}(A) + i[H_{\mathcal{S}}(s) + \lambda(s)V, A]), \quad \tau_T^0(A) = A.$$

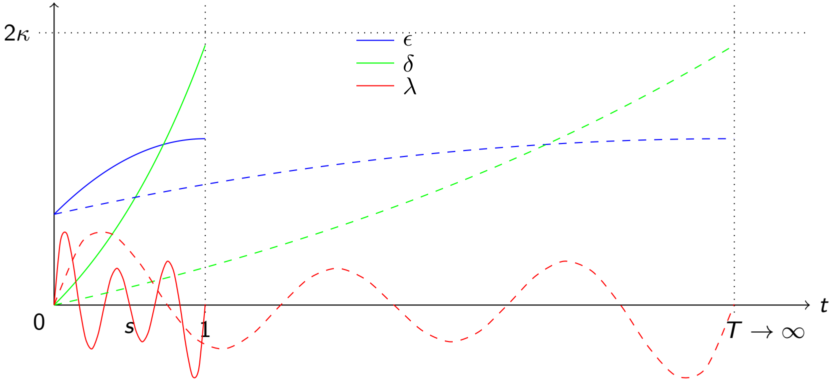
- At fixed epoch $\forall A \in \mathcal{O}$:

$$\partial_t \tau_s^t(A) = \tau_s^t(\delta_{\mathcal{R}}(A) + i[H_{\mathcal{S}}(s) + \lambda(s)V, A]), \quad \tau_s^0(A) = A.$$

- Instantaneous equilibrium assumption:

$$\rho_s^{(L)} = \frac{e^{-\beta H^{(L)}(s)}}{\mathrm{tr}(e^{-\beta H^{(L)}(s)})} \quad \xrightarrow[L\to\infty]{} \quad \rho_s \text{ the unique } (\beta, \tau_s)\text{–KMS state.}$$

The adiabatic limit: $T \to \infty$



The adiabatic limit for KMS states

Definition (Ergodicity)

The state ρ_s is τ_s–ergodic if for all $A, B \in \mathcal{O}$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \rho_s(B^* \tau_s^t(A) B) dt = \rho_s(B^* B) \rho_s(A).$$

Theorem (Abou–Salem, Fröhlich '05; Jakšić, Pillet '14)

If $H_{\mathcal{S}}$ and λ are $C^1([0,1])$ and for a.e. $s \in [0,1]$, ρ_s is τ_s–ergodic, then

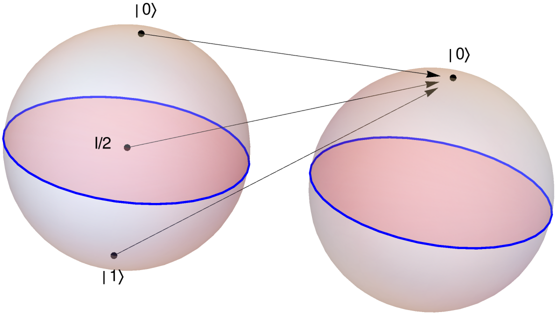
$$\lim_{T\to\infty} \sup_{s\in[0,1]} \|\rho_0 \circ \tau_T^s - \rho_s\| = 0.$$

Proof.

- GNS representation of thermal states + Araki's perturbation theory.
- Avron–Elgart gapless adiabatic theorem.

□

Adiabatic erasure



Assume $\lambda(0) = \lambda(1) = 0$, $H_{\mathcal{S}}(0) = \epsilon(0) I_2$ and $\delta(1) > 0$. Then $\rho_{\mathrm{i}} = \frac{1}{2} I_2$ and

$$\rho_0 = \frac{1}{2} I_2 \otimes \rho_{\mathcal{R}}, \quad \rho_1 = \rho_{\mathrm{f}} \otimes \rho_{\mathcal{R}}$$

with $\rho_{\mathrm{f}} = \frac{e^{-\beta H_{\mathcal{S}}(1)}}{\mathrm{tr}(e^{-\beta H_{\mathcal{S}}(1)})} > 0$.

Saturation of Landauer's bound

- Average total work: $\langle \Delta W \rangle_T = \int_0^1 \rho_0 \circ \tau_T^s(\dot{H}_{\mathcal{S}}(s) + \dot{\lambda}(s)V)ds$.
- Average bath heat: $\langle \Delta Q \rangle_T = -\int_0^1 T\lambda(s)\rho_0 \circ \tau_T^s(\delta_{\mathcal{R}}(V))ds$.
- Memory energy: $\langle \Delta E_{\mathcal{S}} \rangle_T = \rho_0 \circ \tau_T^1(H_{\mathcal{S}}(1)) - \rho_0(H_{\mathcal{S}}(0))$.

The first law of thermodynamics follows:

$$\langle \Delta Q \rangle_T = \langle \Delta W \rangle_T - \langle \Delta E_{\mathcal{S}} \rangle_T.$$

Theorem (Jakšić, Pillet '14)

$$\lim_{T\to\infty} \langle \Delta Q \rangle_T = \beta^{-1} \ln 2 - S(\rho_f)$$

with $S(\rho_f) = -tr(\rho_f \ln \rho_f)$.

Proof.

- $\lim_{T\to\infty} \langle \Delta W \rangle_T = \Delta F = -\beta^{-1} \ln \mathrm{tr}(e^{-\beta H_{\mathcal{S}}(1)}) - \epsilon(0) + \beta^{-1} \ln 2$
- $\lim_{T\to\infty} \langle \Delta E_{\mathcal{S}} \rangle_T = \Delta F + S(\rho_f) - \beta^{-1} \ln 2$.

□

Heat and Work full statistic ($L < \infty$)

Two time measurement:

$$\rho_0 \vdash \xrightarrow{\quad \tau_T^s \quad} T/1 \quad t/s$$

$\uparrow$ $H^{(L)}(0),\ H^{(L)}(0)$ $\qquad\qquad$ $\uparrow$ $H^{(L)}(0),\ H^{(L)}(1)$

- Total work full statistic characteristic function:

$$\begin{aligned}\chi^{(L)}_{W,T}(\alpha) =& \mathrm{tr}\Big(e^{i\alpha H^{(L)}(1)} U^{(L)}_T(s) e^{-i\alpha/2 H^{(L)}(0)} \rho^{(L)}_0 e^{-i\alpha/2 H^{(L)}(0)} U^{(L)*}_T(s)\Big) \\ =& \mathrm{tr}(\rho_1^{(L)}{}^{-i\alpha/\beta} \rho_T^{(L)}(s)^{1+i\alpha/\beta}) \times e^{i\alpha\Delta F} \\ =& S_{-i\alpha/\beta}(\rho_1^{(L)}|\rho_T^{(L)}(1)) \times e^{i\alpha\Delta F}\end{aligned}$$

- Bath heat full statistics: $H^{(L)}(0) = H^{(L)}_{\mathcal{R}} + cste.$ implies

$$\chi^{(L)}_{Q,T}(\alpha) = S_{-i\alpha/\beta}(\rho_0^{(L)}|\rho_T^{(L)}(1)).$$

Heat and Work full statistic in the adiabatic limit

Assumption: Exponential of Rényi relative entropies accept a thermodynamical limit expressed using relative modular operators.

- $\chi_{W,T}(\alpha) = S_{-i\alpha/\beta}(\rho_1|\rho_0\circ\tau^1_T)\times e^{i\alpha\Delta F} = \langle\Omega_{\rho_0\circ\tau^1_T}|\Delta^{-i\alpha/\beta}_{\rho_1|\rho_0\circ\tau^1_T}\Omega_{\rho_0\circ\tau^1_T}\rangle\times e^{i\alpha\Delta F}$.
- $\chi_{Q,T}(\alpha) = S_{-i\alpha/\beta}(\rho_0|\rho_0\circ\tau^1_T) = \langle\Omega_{\rho_0\circ\tau^1_T}|\Delta^{-i\alpha/\beta}_{\rho_0|\rho_0\circ\tau^1_T}\Omega_{\rho_0\circ\tau^1_T}\rangle$.

Theorem (B., Fraas, Jakšić, Pillet '16)

The total work and the bath heat full statistic converge weakly in the adiabatic limit.

$$\lim_{T\to\infty}\chi_{W,T}(\alpha) = e^{i\alpha\Delta F}$$

and

$$\lim_{T\to\infty}\chi_{Q,T}(\alpha) = S_{-i\alpha/\beta}(\rho_0|\rho_1) = 2^{i\alpha/\beta}tr(\rho_f^{1+i\alpha/\beta}).$$

Proof.

The norm convergence of the states in the adiabatic limit implies the strong resolvent convergence of the corresponding relative modular operator.
The results follow from the triviality of the relative modular operator kernel. □

Heat and Work full statistic in the adiabatic limit

Consequences:

- The total work converges in distribution to the free energy variation as expected for classical thermodynamical isothermal quasi–static processes.

$$\Delta W = \Delta F.$$

- The bath heat variation: Let $\rho_{\mathrm{f}} = p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|$.

$$\mathbb{P}_Q(\Delta Q = Q_1) = p = \frac{1}{1 + e^{2\beta\delta(1)}}, \mathbb{P}_Q(\Delta Q = Q_0) = (1-p)$$

  with

$$Q_1 = \beta^{-1}\ln 2 - \beta^{-1}\ln(1 + e^{2\beta\delta(1)}) \text{ and } Q_0 = \beta^{-1}\ln 2 - \beta^{-1}\ln(1 + e^{-2\beta\delta(1)}).$$

Perfect erasure limit

From [Reeb, Wolf '14; Jakšić, Pillet '14]:

$$\lim_{S(\rho_{\mathcal{S},T})\to 0} \langle \Delta Q \rangle_T = +\infty$$

but

$$\lim_{S(\rho_{\mathrm{f}})\to 0} \lim_{T\to\infty} \langle \Delta Q \rangle_T = \beta^{-1} \ln 2.$$

Proposition (B., Fraas, Jakšić, Pillet '16)

- $\langle (\Delta Q - \beta^{-1} \ln 2)^n \rangle = O(p(\ln p)^n)$, *so,*

$$\lim_{\delta(1)\to\infty} \lim_{T\to\infty} \Delta Q = \beta^{-1} \ln 2 \textit{ in } L^n\text{–}\textit{norm.}$$

- $\lim_{\delta(1)\to\infty} \lim_{T\to\infty} \ln \chi_{Q,T}(i\alpha) = \begin{cases} -\frac{\alpha}{\beta} \ln 2 & \textit{if } \alpha < \beta \\ 0 & \textit{if } \alpha = \beta \\ \infty & \textit{if } \alpha > \beta \end{cases}$

Thank you.