SLOW OR NO HEATING

IN MANY-BODY QUANTUM SYSTEMS

FRANÇOIS HUVENEERS (PARIS 9)

joint work with

DMITRY ABANIN,
WOJCIECH DE ROECK,
WEN WEI HO.

Aim: To describe quantum many-body systems with breakdown or quasi-breakdown (i.e. for a very long transient time) of ergodicity.

- Mostly driven systems
- Emergence of (quasi)-conserved quantities
- "Prethermalization"
Example: Ping-Pong

Elastic collisions with the wall and paddle (single particle classical problem)

Two extreme cases:

(a) The paddle moves randomly:
- The ball absorbs energy forever $E(t) \sim t$.
- "Heating up to infinite temperature"

(b) The paddle moves periodically
- The velocity of the ball stays bounded (if driving is smooth)
  $|v(t)| \leq C \forall t$.
- Emergence of an effective conserved quantity.

(See Dolgopyat, De Simoi)
**Question:** Can we find a similar dichotomy in many-body quantum systems?

**Plan of the talk:**

- Basic intuition on this problem (following D'Alessio-Polkovnikov '12 and subsequent criticism)
- Particular case of many-body localized (MBL) systems
- Driven ergodic systems with quasi-conserved quantities
- Generalization to non-driven systems

**Main mathematical tool:**

Successive canonical transformations for many-body systems (cfr. Imbrie '14)
**Hamiltonian:**

\[ H(t) = H^{(0)} + g \ V(t), \quad \text{with} \quad \phi(t) \text{ like} \]

\[ V(t+T) = V(t) \]

\[ \phi(t) \]

(SWITCH BETWEEN TWO HAMILTONIANS)

\[ H^{(0)} \text{ AND } \overline{V} \text{ ARE SUMS OF LOCAL TERMS:} \]

\[ H^{(0)} = \sum_{i \in \Lambda} H^{(0)}_i, \quad \overline{V} = \sum_{i \in \Lambda} \overline{V}_i. \]

E.g. 1-d SPIN CHAIN

\[ H^{(0)} = \sum_{i \in \Lambda} h_i \ \sigma_i^{(z)} + J_i^{(x)} \ \sigma_i^{(x)} \ \sigma_{i+1}^{(x)} \]

\[ \overline{V} = \sum_{i \in \Lambda} J_i^{(z)} \ \sigma_i^{(z)} \ \sigma_{i+1}^{(z)} \]

\[ i \in \Lambda, \ \ \Delta \rightarrow \infty \]
How to define heating up to $T = \infty$?

Via Floquet Theory:

- Unitary Dynamics: $|\psi(t)\rangle = U(t) |\psi(0)\rangle$
- Define an effective Hamiltonian:
  $U(t) = e^{-iH_{\text{eff}} t}$
  (Always possible at $|\Lambda| < \infty$)

2 cases as $\Lambda \to \infty$

(a) $H_{\text{eff}}$ is a sum of local terms (or quasi-local terms):

$H_{\text{eff}} = \sum_{i \in \Lambda} H_{\text{eff},i}$

$\Rightarrow$ behaves well as $\Lambda \to \infty$

$\Rightarrow$ effective conserved quantity

$\Rightarrow$ bound on heating up.

(b) $H_{\text{eff}}$ is not a sum of local terms

$\Rightarrow$ no limit as $\Lambda \to \infty$

$\Rightarrow$ system goes towards infinite temperature state = maximal entropy.
Settle the issue via an explicit expansion?

\[ e^A e^B = e^{A+B + \frac{1}{2}[A,B] + \ldots} \]

Here:

\[ e^{-i \frac{T}{2} H^{(0)}} e^{-i \frac{T}{2} (H^{(0)} + g \overline{V})} = e^{-i T H_{\text{eff}}} \]

\[ \Rightarrow H_{\text{eff}} = \left( H^{(0)} + \frac{g}{2} \overline{V} \right) + T \frac{i g}{4} \left\{ H^{(0)}, \overline{V} \right\} + O(T^2) \]

Naive hope: \( \tilde{E} \): energy scale

(a) \( T \tilde{E} << 1 \) (high frequency): expansion converges

(b) \( T \tilde{E} \gtrsim 1 \) (low frequency): expansion diverges

Remark: consistent with intuition from forced pendulum:

\[ \ddot{\theta} = -2^2 \sin \theta + g \sin (\theta - \omega t) \]

\[ \left| \text{Stochastic layer} \right| \sim g e^{-\frac{|\omega|}{2}} \]
Free Systems

(Say $H(t) = \Gamma (\mathcal{R}(t))$)

In this case, this is correct!

Why?

Energy absorption remains bounded if

$$\omega = \frac{2\pi}{\Gamma} > W$$

Energy absorption rate:

$$\Gamma^\text{ii}(\omega) \sim g^2 \sum_{\eta_1, \eta_1'} e^{-\beta E_\eta} |\langle \eta_1' | \mathcal{V}_i | \eta_1 \rangle|^2 \delta(E_{\eta_1} - E_{\eta_1'} - \omega)$$

(linear response in $g = \text{Fermi Golden Rule}$)

"System cannot absorb a photon with frequency $\omega > W$".

What about more generic systems?
In general, one expects BHC expansion to be only asymptotic at small $t$.

Why?

Let us consider again

$$
\Gamma_p^{ii}(\omega) \sim g^2 \sum_{\eta, \eta'} e^{-\beta E_\eta} |\langle \eta | V_i | \eta' \rangle|^2 \delta(E_\eta - E_{\eta'} - \omega)
$$

(Now $|\eta', \eta\rangle$ are many-body eigenstates!)

Trick:

$$
\langle \eta' | V_i | \eta \rangle = \frac{\langle \eta' | [\mathcal{H}, V_i] | \eta \rangle}{E_{\eta'} - E_{\eta}}
$$

$$
= ... = \frac{\langle \eta' | [\mathcal{H}, \ldots, \mathcal{H}, V_i \ldots, J] | \eta \rangle}{(E_{\eta'} - E_{\eta})^n}
$$

Then:

$$
| [\mathcal{H}, \ldots, \mathcal{H}, V_i \ldots, J] | \leq e^n n!
$$

So:

For $E_{\eta'} - E_{\eta} = \omega$ and $\frac{\omega}{\varepsilon} \gg 1$:

$$
\int dw \, \Gamma_p^{ii}(\omega) \sim \frac{n!}{(\omega/\varepsilon)^n} \sim e^{-\omega/\varepsilon}
$$

Optimization: $n \sim \frac{\omega}{\varepsilon}$
Remark 1

For systems satisfying Eigenstate Thermalization Hypothesis (ETH), one expects no better bound:

\[ |\langle \eta' | \mathcal{U} | \eta \rangle|^2 \sim \frac{f(E_{\eta'}/E_{\eta})}{\lim(21)} \]

Remark 2

In \( d=1 \), Araki analyticity allows to improve the exponential bound into something super-exponential.

See our paper PRL 115 for more details.

Let us now look for a better technology than BHC!
Rotating Frame

Unitary Evolution for General Time $t$:

$$U(t) = P(t) e^{-i\text{Heff} t}$$

Periodic: $P(t+T) = P(t)$

and Unitary

$P(t)$: Rotating Frame Transformation

Solves the Equation:

$$P^+(t) \left( H(t) - i \frac{d}{dt} \right) P(t) = \text{Heff}$$

Better way to get $\text{Heff}$

• Analogy with Time-Independent Set-up:
  Find $U$ so that $U^+ H U = \text{Diagonal}$. (See Inbrie '74)

• Here: Find $P(t)$ periodic so that $\text{Heff}$ is very time-independent

We now see three examples:
I. QUASI-CONSERVED EXTENSIVE QUANTITIES

- REMIND LINEAR RESPONSE PREDICTION:

\[ \Gamma(\omega) \sim -\frac{\omega}{\epsilon} \frac{1}{\omega} \quad \omega > 0, \]

FOR GENERAL LATTICE HAMILTONIANS WITH BOUNDED ON-SITE ENERGY \( \epsilon \).

- WE WANT TO EXTEND THIS RESULT BEYOND LINEAR RESPONSE REGIME \( |\epsilon| > 0 \).

- WE TAKE

\[ H = H^{(0)} + g \varphi(\omega t), \quad \bar{V} \]

\[ g \ll \epsilon, \quad g \ll \omega, \quad \frac{1}{T} \int_0^T \varphi(\omega s) \, ds = 0 \]

\[ \text{ENERGY PER SITE IN } H^{(0)}. \]

- TAKE \( P(t) \) OF THE FORM

\[ P(t) = e^{\frac{g}{3} A(\hbar \omega t)} \]

\[ A(\omega t) = \sum_i A_i(\omega t) \]

IN

\[ P^+(t) \left\{ \frac{H(t) - i}{d^\hbar} \int P(t) \right\} \sim H^{\text{eff}} \]

WE WILL NOT REACH EQUALITY!
Important: Such $P(t)$ preserves quasi-locality: if $O_i$ is local on site $i \in \Lambda$, then $P^+(t) O_i P(t)$ is quasi-local around site $i \in \Lambda$:

$$P^+(t) O_i P(t) = \sum_{B \ni i} \tilde{O}_B$$

with $\| \tilde{O}_B \| \leq \left( \frac{g}{\omega} \right)^{|B|}$.

How to select $A(\omega t)$?

Expand in $\frac{g}{\omega}$ and impose that the highest order term that depends on $t$ vanishes:

$$P^+(t) \left\{ H(t) - i \frac{d}{dt} \right\} P(t) \quad = \quad H^{(0)} + g \phi(\omega t) \overrightarrow{V} + O\left( \frac{g}{\omega} \cdot \xi \right)$$

$$\quad \quad \quad - i g \frac{dA}{dt}(\omega t) + O\left( \frac{g}{\omega} \cdot g \right)$$

Cancel this

$$\Rightarrow A(\omega t) = \left( \int_0^{\omega t} d\omega \phi(\omega) \right) \cdot \overrightarrow{V}$$

Periodic since $\frac{1}{T} \int_0^T d\omega \phi(\omega) = 0$. 

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Upshot: In the rotating frame, we have a new Hamiltonian with smaller time-dependent part:

\[ g \rightarrow \text{max} \left\{ \frac{\mathcal{g}}{\omega} \epsilon, \frac{\mathcal{g}}{\omega} \mathcal{g} \right\} \ll g \]

We iterate this procedure \( n \) times.

1. We gain powers of \( \mathcal{g}/\omega \) \( \Rightarrow \)

\[ \mathcal{g} \xrightarrow{\text{max}} \frac{n}{\omega} \left\{ \left( \frac{\mathcal{g}}{\omega} \right)^n \epsilon, \left( \frac{\mathcal{g}}{\omega} \right)^n \mathcal{g} \right\} \]

2. We have to stop at some optimal \( n \) as terms become less and less local.

Result:

We obtain a quasi-conserved quantity up to very long (quasi-) exponential times in \( \omega/\epsilon \). We call it \( \tilde{H}_{\text{eff}} : \)

\[
\begin{aligned}
\frac{1}{|N|} \left\| U^+(kt) \tilde{H}_{\text{eff}} U(kt) - \tilde{H}_{\text{eff}} \right\| &
\leq C.(kt) \cdot \epsilon - \frac{\omega}{\epsilon} \frac{1}{1 + \log \frac{1}{\epsilon}} \\
\tilde{H}_{\text{eff}} &= \sum_i \tilde{H}_{\text{eff},i}
\end{aligned}
\]
II. Hubbard Model (No Driving)

Fermions on the lattice with

\[ H = U \sum_i n_{i\uparrow} n_{i\downarrow} + t \sum_{\langle i,j \rangle, \sigma} a_{i\sigma}^\dagger a_{j\sigma} \]

Interaction \quad Hopping

We consider the regime

\[ \frac{U}{t} \gg 1 \]

Particles come in singleons/doublon:

Very hard to create/destroy doublon:

Big energy mismatch!

\[ \Delta E = U \gg t \]
True conserved quantities:

- Energy
- Number of fermions with spin $\uparrow$
- Number of fermions with spin $\downarrow$

Quasi-conserved quantity:

Dressed version of number of doublons for exponential times in $\frac{U}{t}$.

$\tilde{N}_D$ is such that $\tilde{N}_D = \sum_i \tilde{N}_{D,i}$

And

$$\frac{1}{|\tilde{N}|} \| U^+(t) \tilde{N}_D U(t) - \tilde{N}_D \| \leq C \cdot t \cdot e^{-\frac{U}{t} \cdot \frac{1}{1+\frac{t}{U}}}$$

$t$ = Hopping parameter here!

We adapt our previous method:

We write:

$$H = U \cdot N_D + t \cdot H_{\text{pre}} + t \cdot H_{\text{des}}$$

Number of doublons

Hopping preserving $N_D$

Hopping creating or destroying doublons

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TRY TO GET RID OF $H_{\text{loc}}$

VIA CANONICAL TRANSFORMATIONS THAT PRESERVE LOCALITY:

TAKE $P = e^{\frac{t}{u}A}$, $A = \sum_{i} A_{i}$

THEN

$$P^+ HP = H + \frac{t}{u}[H, A] + (\frac{t}{u})^{2} [A, [A, H]] + \ldots$$

$$= U N_{D} + t H_{\text{me}} + \frac{t}{u} H_{\text{loc}}$$

$$+ \frac{t}{u} [N_{D}, A] + O(\frac{t}{u}, t)$$

SELECT $A$ TO CANCEL

$$[A, N_{D}] = H_{\text{loc}}$$

$$\Rightarrow \langle \eta' | A | \eta \rangle = \frac{\langle \eta' | H_{\text{loc}} | \eta \rangle}{N_{D}(\eta) - N_{D}(\eta')}$$

ALWAYS $\neq 0$ WHEN $\langle \eta' | H_{\text{loc}} | \eta \rangle \neq 0$

$|\eta\rangle$: CLASSICAL CONFIGURATIONS

$A = \sum_{i} A_{i}$ BECAUSE $H_{\text{loc}} = \sum_{i} H_{\text{loc}, i}$

ITERATE THIS PROCEDURE
III. MBL SYSTEMS (WITH DRIVING)

Many-body localized (MBL) systems have eigenstates that do not obey ETH

\[ \Rightarrow \text{some previous argument does not apply} \]

\[ \Rightarrow \text{what is the behavior of MBL system under driving?} \]

Very short intro to MBL:

- Analogous of Anderson localization for interacting many-body systems
- Complete lack of thermalization

\[ \begin{align*}
\text{COLD} & \quad \text{HOT} \\
\text{COLD} & \quad \text{HOT}
\end{align*} \quad \begin{align*}
t = 0 \\
t = \infty
\end{align*} \]

*Typical example: 1d spin chain

\[ H = \sum_i \left( h_i \sigma_i^{(z)} + J^{(2)} \sigma_i^{(z)} \sigma_{i+1}^{(z)} + J^{(x)} \sigma_i^{(x)} \right) \]
DEFINITION VIA EIGENSTATES

UNITARY $U$ s.t. $U^* H U = \Delta$ (DIAGONAL)

"$H$ IS MBL $\iff U$ PRESERVES LOCALITY"

PRESERVING LOCALITY MEANS

$O_i$ LOCAL AT $i \in \Lambda$ $\implies$

$U^* O_i U = \sum_{A \in \Lambda} \tilde{O}_A$, $\|O\|_{\Lambda} \leq C e^{-\frac{|A|}{3}}$

WITH $\frac{3}{3}$: LOCALIZATION LENGTH

LOCAL INTEGRALS OF MOTION

(ABANIN, HUSE, OGANESYAN...)

$\Delta = \sum_i C_i \tau_i^2 + \sum_{i<j} C_{ij} \sigma_i^2 \sigma_j^2 + \sum_{i<j<k,l} (...) + ...$

$\tilde{L}_i^2 = U \sigma_i^2 U^*$ (LIOMS)

$H = \sum_i C_i \tilde{L}_i^2 + \sum_{i<j} C_{ij} \tilde{L}_i^2 \tilde{L}_j^2 + \sum_{i<j<k,l} (...) + ...$

$|C_{i_1, ..., i_m}| \leq C e^{-\frac{|i_m-i_1|}{3}}$

$\implies$ DYNAMICS = COLLECTION OF INDEPENDENT QUASI-LOCAL PSEUDO-SPINS ($\tilde{L}_i^2$)
Driven MBL Systems

\[ H(t) = H(0) + g \int_0^T V(t) \]

\[ V(t+T) = V(t) \quad \text{AND} \]

\[ \frac{i}{T} \int_0^T d\tau V(\tau) = 0. \]

Same question as before: can we find \( \mathcal{H}_{\text{eff}} = \sum_i \mathcal{H}_{\text{eff},i} \quad \text{s.t.} \]

\[ P_+(t) \lessgtr \left\{ H(t) - i \frac{d}{dt} \right\} P(t) = \mathcal{H}_{\text{eff}}? \]

Perhaps yes! Why?

Due to Lions structure, it is no longer true that the system can absorb any frequency by a local move:

MBL material cannot act as bath!

Result (non-mathematical)

\( \mathcal{H}_{\text{eff}} \) exists and is itself MBL if

\[ \frac{g}{w} \ll 1, \quad \frac{g^2}{w W} \ll 1 \]

Disorder strength
Remark 1

Example of breakdown of linear response:

Mott law (or analogous many-body counter part) predicts

\[ \tau(\omega) \sim \omega^2 \]

Non-equilibrium phenomenon

- Transient regime where LR applies: system heats up a bit
- Next, system saturates and reaches non-equilibrium state: LR not valid anymore

Remark 2

Our conditions allow \( \tau \gg \omega \)

\( \Rightarrow \) For most individual times, \( H(t) \) may not be valid
REMARK 3

How to get the result?

Canonical transformations again!

\[ P^+(t) \gamma H(t) - i \frac{d}{dt} \gamma P(t) = H_{\text{eff}} \]

Difference with previous

- \( H^{(0)} \) is local in eigenstate basis
- \( \Rightarrow \) allows to cancel more terms at once
- \( \Rightarrow \) quadratic convergence (à la KAM)
- \( \Rightarrow \) allows to overcome the problem of too many commutators.