Decay of correlations and absence of superfluidity in the disordered Tonks-Girardeau gas

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based on joint works with:

R. Seiringer (to appear in: *New Journal of Physics* arXiv:1512.05282)

R. Sims (to appear in: *Commun. Math. Phys.* arXiv:1509.00450)

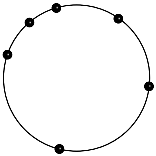
Grenoble, February 3, 2016

Outline

I. The Tonks-Girardeau gas in disorder

Consider N Bosons on a ring of length L

with hard-core repulsion



$$\mathcal{H}_{L,\omega} = \sum_{j=1}^{N} \left(H^{+}_{L,\omega}\right)_j + g \sum_{1\leq j<k\leq N} \delta(x_j - x_k)\,,$$

and random one-particle Hamiltonian $H^{+}_{L,\omega} = -\frac{d^2}{dx^2} + V_\omega(x)$ on $L^2([0,L])$
with periodic bc.

Tonks-Girardeau limit: $g \to \infty$

Normalized eigenfunctions in the Tonks-Girardeau limit:

$$\Psi(x_1,\ldots,x_N) = \frac{1}{\sqrt{N!}} \det\left(\varphi^{\sharp_N}_{j_\alpha,L}(x_\beta)\right)_{\alpha,\beta=1}^{N} \prod_{1\leq j<k\leq N} \operatorname{sign}(x_j - x_k)$$

where $(\varphi^{\sharp_N}_{j_\alpha,L})$ are norm. eigenfunctions of $H^{\sharp_N}_{L,\omega}$ with:

- periodic bc ($\sharp_N = 1$) in case N odd, Girardeau '60
- anti-periodic bc ($\sharp_N = -1$) in case N even. Lieb-Liniger '62

One-particle reduced density matrix

For any eigenfunction Ψ the kernel of **one-particle reduced density matrix** ($0 \leq \gamma_\Psi \leq N$, $\operatorname{Tr} \gamma_\Psi = N$) takes the form of a determinant:

$$\gamma_\Psi(x,y) := N \int \Psi(x, x_2, \ldots, x_N) \overline{\Psi(y, x_2, \ldots, x_N)} \, dx_2 \ldots dx_N$$

$$= \det \begin{pmatrix} 0 & \varphi^{\sharp N}_{j_1,L}(x) \cdots \varphi^{\sharp N}_{j_N,L}(x) \\ \hline \varphi^{\sharp N}_{j_1,L}(y) & \\ \vdots & K_N(x,y) \\ \hline \varphi^{\sharp N}_{j_N,L}(y) & \end{pmatrix}$$

with $[K_N(x,y)]_{\alpha,\beta} := \delta_{\alpha,\beta} - 2 \int_{[x,y]} \varphi^{\sharp N}_{j_\alpha,L}(z) \overline{\varphi^{\sharp N}_{j_\beta,L}(z)} dz$ for all $x \leq y$.

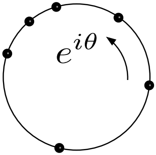
- Only the density coincides with that of non-interacting Fermions: $\varrho_\Psi(x) = \gamma_\Psi(x,x) = \sum_{\alpha=1}^N |\varphi^{\sharp N}_{j_\alpha,L}(x)|^2$
- **BEC** refers to a macroscopic value of $\|\gamma_\Psi\|_\infty$.

  Without disorder ($V_\omega = 0$): Lenard '64

  $\gamma_\Psi(x,y) \sim |x-y|^{-1/2}$ i.e. **quasi-condensation:** $\|\gamma_\Psi\|_\infty \sim N^{1/2}$

Superfluid density (aka: stiffness): grand-canonical definition

Ground-state energy shift under twisting the bc $e^{i\theta}$ as one particle moves around the ring:



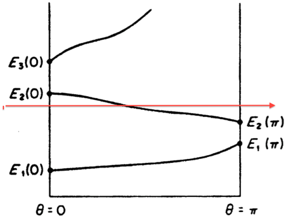
$$\rho_s := \limsup_{\theta\to 0} \frac{1}{\theta^2} \limsup_{L\to\infty} L\left(E_L(N_\mu,\theta) - E_L(N_\mu,0)\right)$$

with $E_L(N,\theta)$ ground-state energy with bc

$$\Psi(x_1,\dots,x_j+L,\dots,x_N) = e^{i\theta}\,\Psi(x_1,\dots,x_j,\dots,x_N) \quad \text{for all } j.$$

The ground-state energy for fixed chemical potential μ:

$$N_\mu := \min\{N_\mu^+, N_\mu^-\},$$
$$N_\mu^\pm := \operatorname{Tr} 1_{(-\infty,\mu]}(H_L^\pm)$$



Band structure of the one-particle operators $H^\theta_{L,\omega}$. (cf. Reed/Simon IV)

- Without disorder ($V_\omega = 0$): $\rho_s = \lim_{L\to\infty} \frac{1}{L} N_\mu = \frac{\sqrt{[\mu]_+}}{2\pi^2}$

Localization hypothesis on one-particle operator $H^{\pm}_{L,\omega}$ (ECL in $J \subset \mathbb{R}$)

There exist $C, \ell \in (0, \infty)$ and $\xi \in (0, 1]$ such that for all $1 \leq n, m \leq L$ and all $L \in \mathbb{N}$

$$\boxed{\mathbb{E}\left[Q^{\pm}_{L}(n, m; J)\right] \leq C \exp\left(-\frac{\mathrm{dist}(n, m)^{\xi}}{\ell^{\xi}}\right)}$$

where $\mathrm{dist}(\cdot, \cdot)$ denotes the Euclidean distance on the torus and

$$Q^{\pm}_{L}(n, m; J; \omega) := \sum_{j,\, E^{\pm}_{j,L} \in J} \Phi^{\pm}_{j,L}(n; \omega)\Phi^{\pm}_{j,L}(m; \omega)\,,$$

with $\Phi^{\pm}_{j,L}(n; \omega) := \left(\int_{I_n} \left|\varphi^{\pm}_{j,L}(x; \omega)\right|^2 dx\right)^{\frac{1}{2}}$ and $I_n := [n, n + 1)$.

- Generically satisfied for random $H^{\pm}_{L,\omega} = -(\frac{d^2}{dx^2})^{\pm} + V_{\omega}(x)$ in one dimensions for any $J \subset \mathbb{R}$ which is bounded from above.
- Implies dynamical localization.

Immediate consequence of ECL

Given the dynamics of **free fermions** $\Psi(x_1,\dots,x_N) = \frac{1}{\sqrt{N!}} \det\left(\varphi_{j_\alpha,L}(x_\beta)\right)_{\alpha,\beta=1}^{N}$, and the corresponding one-particle density

$$\varrho_t(x) := \Gamma_{\Psi_t}(x,x) = \left(e^{-itH_{L,\omega}}\Gamma_\Psi e^{itH_{L,\omega}}\right)(x,x)\,.$$

(ϱ_t agrees with density of the time-evolved Tonks-Girardeau gas!)

Theorem

If the range of Γ_Ψ at $t=0$ falls within a regime $J \subset \mathbb{R}$ of (ECL), then there exists an $A \in (0,\infty)$ which is independent of N, L such that

1. *the* **total number of particles** *on any subset $I \subset [0,L]$* **changes on average by order one** *only:*

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}}\left|\int_I \varrho_t(x)dx - \int_I \varrho_0(x)dx\right|\right] \le A \quad \text{for all } I \subset [0,L]\,,$$

2. *for any pair of subsets $I \subset K \subset [0,L]$:*

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}}\int_I \varrho_t(x)dx\right] \le \mathbb{E}\left[\int_K \varrho_0(x)dx\right] + A\exp\left(-\frac{\operatorname{dist}(I,K^c)^\xi}{\ell^\xi}\right)\,.$$

Immediate consequence of ECL

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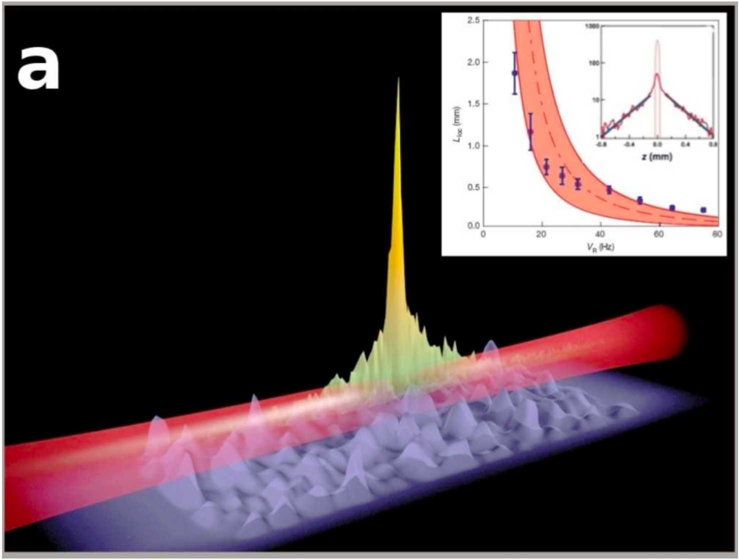
- One facet of **many-body localization (MBL)** – open problem for $g < \infty$!

Direct observation of Anderson localization of matter-waves in a controlled disorder



Billy, Josse, Zuo, Bernard, Hambrecht, Lugan, Clément, Sanchez-Palencia, Bouyer, Aspect: Nature **453** (2008) 891-894

Sketch of proof of MBL for free Fermions

By assumption on the range of the initial state, we have $\Gamma_{\Psi_t} = U_t^* \Gamma_\Psi U_t$ with $U_t = e^{itH_L} P_J(H_L)$. Consequently,

$$\begin{aligned}\left| \sum_{x\in I} \varrho_t(x) - \sum_{x\in I} \varrho_0(x) \right| &= \left| \mathrm{Tr}\, 1_I U_t^* \Gamma_\Psi U_t - \mathrm{Tr}\, 1_I \Gamma_\Psi \right| \\ &= \left| \mathrm{Tr}\, 1_I U_t^* 1_{I^c} \Gamma_\Psi U_t - \mathrm{Tr}\, 1_{I^c} U_t^* 1_I \Gamma_\Psi U_t \right| \\ &\le \| 1_I U_t^* 1_{I^c} \|_1 + \| 1_{I^c} U_t^* 1_I \|_1 \,,\end{aligned}$$

Hence:

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \| 1_{I^c} U_t^* 1_I \|_1 \right] \le \sum_{\substack{I_n \cap I \neq \emptyset \\ I_m \cap I^c \neq \emptyset}} \mathbb{E}\left[\sup_{t\in\mathbb{R}} \| 1_{I_m} U_t^* 1_{I_n} \|_1 \right] \,,$$

and the right side is bounded by a constant on account of dynamical localization.

II. Main Results

Theorem (Absence of ODLRO – Hilbert-Schmidt norm) (Seiringer/W. '15)

For any many-particle eigenstate Ψ*, which is composed of a selection one-particle states* $\{\varphi^{\sharp N}_{j_\alpha,L}\}_{\alpha=1}^N$ *corresponding to an energy regime* J*, If condition (ECL) holds for* J*, then there exist* $A \in (0,\infty)$ *independent of* L *and* N *such that*

$$\mathbb{E}\left[\|1_{I_n}\gamma_\Psi 1_{I_m}\|_2^\sigma\right] \leq A \exp\left(-\frac{2}{3}(1-\sigma)\frac{\text{dist}(n,m)^\xi}{(2\ell)^\xi}\right)$$

for all $1 \leq n, m \leq L$ *and all* $2/5 \leq \sigma < 1$*.*

- Implies **absence of BEC** provided the local density fluctuations are bounded, i.e., for some $p > 2$:

  $$\sup_{n,L} \mathbb{E}\left[\left(\text{Tr}\, 1_{I_n} P_J(H_L^{\pm})\right)^p\right] < \infty\,.$$

  Then for any sequence Ψ of eigenstates composed of one-particle states $(\varphi^{\sharp N}_{j_\alpha,L})_{\alpha=1,\dots,N}$ whose energies fall into a regime J of (ECL), the almost-sure convergence

  $$\lim_{L\to\infty}\frac{\|\gamma_\Psi\|_\infty}{L^r} = 0$$

  holds for any $\frac{2}{p} < r \leq 1$.

Proof of absence of BEC given no ODLRO

Since $\|\gamma_\Psi\|_\infty \le \max_n \sum_m \|1_{I_n}\gamma_\Psi 1_{I_m}\|_\infty$:

$$\mathbb{E}\left[\|\gamma_\Psi\|_\infty^\sigma\right] \le \sum_n \mathbb{E}\left[\left(\sum_m \|1_{I_n}\gamma_\Psi 1_{I_m}\|_\infty\right)^\sigma\right] \le \sum_n \left(\sum_m \mathbb{E}\left[\|1_{I_n}\gamma_\Psi 1_{I_m}\|_\infty^\sigma\right]^{1/\sigma}\right)^\sigma ,$$

By absence of ODLRO in Hilbert-Schmidt sense:

$$\mathbb{E}\left[\|\gamma_\Psi\|_\infty^\sigma\right] \le CL$$

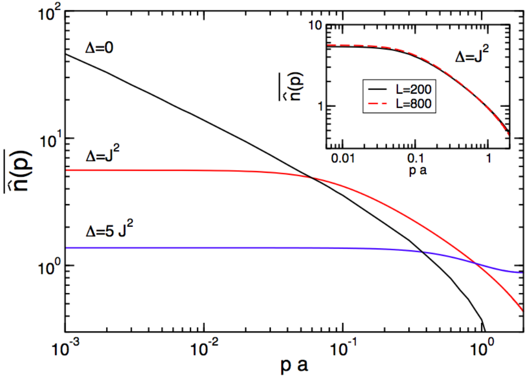
with some $C < \infty$ that is independent of L and N. A Chebychev estimate then implies for any $\varepsilon > 0$ and $r > 0$

$$\mathbb{P}\left(\|\gamma_\Psi\|_\infty > \varepsilon L^r\right) \le \frac{\mathbb{E}\left[\|\gamma_\Psi\|_\infty^\sigma\right]}{\varepsilon^\sigma L^{r\sigma}} \le \frac{C}{\varepsilon^\sigma} L^{1-\sigma r} . \qquad (1)$$

If we choose $r > 2/\sigma$, and the rhs is summable in L. The Borel-Cantelli lemma thus yields the claimed almost-sure convergence. □

Momentum distribution: $n(k) := \frac{1}{L} \iint e^{ik(x-y)} \mathbb{E}\left[\gamma_\Psi(x,y)\right] dxdy$

- remains uniformly bounded: $|n(k)| \leq \sup_L \frac{1}{L} \sum_{n,m} \mathbb{E}\left[\| 1_n \gamma_\Psi 1_{I_m} \|_2\right] < \infty.$
- numerical predictions: De Martino/Thorwart/Egger/Graham '05



(At large values $|k| \to \infty$, one expects an algebraic fall-off $n(k) \sim k^{-4}$.
Olshanii/Dunjko '03, Barth/Zwerger '11)

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- numerical predictions: De Martino/Thorwart/Egger/Graham '05

Non-interacting Bose gas in disorder:

E.g. repulsive Poisson potential: $V_\omega(x) = \sum_j U(x - p_{j,\omega}), \quad U \geq 0$

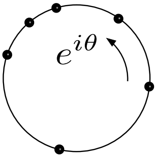
Luttinger/Kac '73/'74, Luttinger/Kac/Sy '73

(see also: Lenoble/Pastur/Zagrebnov '04, Lenoble/Zagrebnov '07)

Due to drastic suppression of the density of states near bottom of the spectrum (aka: *Lifshitz tails*), the critical dimension for the occurrence of BEC is lowered to $d = 1$.

No superfluidity

Ground-state energy shift under twisting the bc $e^{i\theta}$ as one particle moves around the ring:



$$\rho_s := \limsup_{\theta\to 0} \frac{1}{\theta^2} \limsup_{L\to\infty} L\,(E_L(N_\mu,\theta) - E_L(N_\mu,0))$$

- Without disorder ($V_\omega = 0$): $\rho_s = \frac{\sqrt{[\mu]_+}}{2\pi^2}$

Theorem (Absence of superfluidity) (Seiringer/W. '15)

If (ECL) holds for the energy regime $(-\infty,\mu]$, then for any $\theta > 0$ and almost surely:

$$\limsup_{L\to\infty} L\,(E_L(N_\mu,\theta) - E_L(N_\mu,0)) = 0\,.$$

As a consequence, the superfluid density ρ_s is zero almost surely.

IV. Elements of the proof

Proof of decay of correlations is based on:

Lemma (Improved Hadamard bound) (Sims/W. '15)

Let $v, \hat{u} \in \mathbb{C}^p$, $u, \hat{v} \in \mathbb{C}^q$, *and* $K = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in C^{(p+q)\times(p+q)}$ *with* $\|K\| \leq 1$.

Then:

$$\left| \det \begin{pmatrix} \alpha & v^T & u^T \\ \hat{u} & A & B \\ \hat{v} & C & D \end{pmatrix} \right| \leq \sqrt{e}\, \|v\| \|\hat{v}\| \|B\| + |\alpha| + \|u\| \|\hat{u}\| + \|v\| \|\hat{u}\| + \|\hat{v}\| \|u\| \,.$$

By linearity of the determinant, i.e.

$$\det \begin{pmatrix} \alpha & w^T \\ \hat{w} & K \end{pmatrix} = \alpha \det K - \langle w, \mathrm{adj} K\, \hat{w} \rangle$$

and the fact that $\|K\| \leq 1$ it is enough to establish the bound for $\alpha = 0$, $u = 0$ and $\hat{u} = 0$.
Moreover, w.l.o.g. $\|\hat{v}\| = 1$.

- **Apply a unitary operator:**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & 1_q \end{pmatrix} \begin{pmatrix} 0 & v^T & 0 \\ 0 & A & B \\ \hat{v} & C & D \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & U^T & 0 \\ 0 & 0 & 1_q \end{pmatrix} = \begin{pmatrix} 0 & (Uv)^T & 0 \\ 0 & VAU^T & VB \\ \hat{v} & CU^T & D \end{pmatrix} =: M\,.$$

with U taking v into $(0, \ldots, 0, \|v\|)$ and V s.t. VAU^T is upper triangular. Let γ denote the lower right entry of VAU^T.

- **Perform a row operation:** Subtract s times the $(p+1)^{\text{th}}$ row from the first row, for some $s \in \mathbb{C}$.

- **Apply Hadamard inequality:**

$$|\det M| \le \sqrt{|\|v\| - s\gamma|^2 + |s|^2\|B\|^2}\sqrt{|\gamma|^2 + \|B\|^2}\prod_{\alpha=1}^{q}\sqrt{1 + |\hat{v}_\alpha|^2}\,.$$

and use

$$\prod_{\alpha=1}^{q}\sqrt{1 + |\hat{v}_\alpha|^2} = \exp\left(\frac{1}{2}\sum_{\alpha=1}^{q}\ln\left(1 + |\hat{v}_\alpha|^2\right)\right) \le \exp\left(\frac{1}{2}\sum_{\alpha=1}^{q}|\hat{v}_\alpha|^2\right) = \sqrt{e}\,,$$

- **Optimize:** $s = \overline{\gamma}\,\|v\|(|\gamma|^2 + \|B\|^2)^{-1}$.

Proof idea of absence of superfluidity

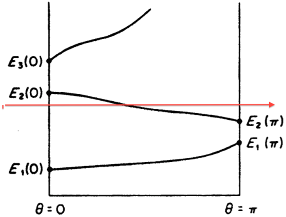
Ground-state energy of the θ-twisted TG gas with N_μ particles:

$$E_L(N_\mu,\theta) = \sum_{j=1}^{N_\mu} E_{j,L}(\theta_\mu)\,, \qquad \theta_\mu := \theta + \frac{\pi}{2}\left(1+(-1)^{N_\mu}\right).$$

Note: $E_{N_\mu,L}(\theta_\mu) \le \mu$

$$N_\mu := \min\{N_\mu^+, N_\mu^-\}$$

$$N_\mu^\pm := \mathrm{Tr}\, 1_{(-\infty,\mu]}(H_L^\pm)$$



The proof of absence of superfluidity is then based on the **variational principle**:

$$\boxed{E_L(N_\mu,\theta) = \mu N_\mu + \inf\left\{\mathrm{Tr}[H_L(\theta_\mu)-\mu]\gamma \,\middle|\, 0\le\gamma\le 1\,,\ \mathrm{Tr}\,\gamma \le N_\mu\right\}\,.}$$

Note it is possible here to relax the condition $\mathrm{Tr}\,\gamma = N_\mu$ to $\mathrm{Tr}\,\gamma \le N_\mu$ exactly because $E_{N_\mu,L}(\theta_\mu) \le \mu$.

Proof idea of absence of superfluidity

Trial density matrix:

$$\gamma = \frac{\tilde{\gamma}}{\max\{\|\tilde{\gamma}\|, 1\}}, \quad \text{with} \quad \tilde{\gamma} := \sum_{j:\, E_{j,L}^{\sharp\mu} \leq \mu} e^{i\psi_{j,L}} |\varphi_{j,L}^{\sharp\mu}\rangle\langle\varphi_{j,L}^{\sharp\mu}| e^{-i\psi_{j,L}} .$$

Note $\mathrm{Tr}\,\tilde{\gamma} = N_\mu$, hence $0 \leq \gamma \leq 1$ and $\mathrm{Tr}\,\gamma \leq N_\mu$.

Trial phase functions $\psi_{j,L} : [0, L] \to \mathbb{R}$ are chosen continuous, piecewise linear such that $\psi_{j,L}(0) = 0$ and $\psi_{j,L}(L) = \theta$:

- Localization hypothesis imlies that there is $\ell \in (0, \infty)$, $\xi \in (0, 1]$ and, for every $L \in \mathbb{N}$, a random amplitude $A_L \geq 0$ that is uniformly integrable, $\sup_{L \in \mathbb{N}} \mathbb{E}\,[A_L] < \infty$, such that

  $$\Phi_{j,L}^{\sharp\mu}(n) \leq A_L\, L^{3/2} \exp\left(-\frac{L^\xi}{(4\ell)^\xi}\right) =: \delta$$

  for all n in an interval $I_j(\delta)$ of length at least $(L-4)/4$.
- Set $\psi_{j,L}$ linear on $I_j(\delta)$ with slope $\theta/|I_j(\delta)|$, and constant otherwise.

Proof idea of absence of superfluidity

Trial density matrix:

$$\gamma = \frac{\tilde{\gamma}}{\max\{\|\tilde{\gamma}\|, 1\}}, \quad \text{with} \quad \tilde{\gamma} := \sum_{j:\, E^{\sharp\mu}_{j,L} \leq \mu} e^{i\psi_{j,L}} |\varphi^{\sharp\mu}_{j,L}\rangle\langle\varphi^{\sharp\mu}_{j,L}| e^{-i\psi_{j,L}} .$$

Note $\mathrm{Tr}\,\tilde{\gamma} = N_\mu$, hence $0 \leq \gamma \leq 1$ and $\mathrm{Tr}\,\gamma \leq N_\mu$.

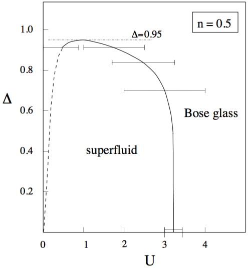
Computations/simple estimates show:

- $$\mathrm{Tr}[H_L(\theta_\mu) - \mu]\gamma \leq -\frac{\mathrm{Tr}[H^{\sharp\mu}_L - \mu]_-}{\max\{\|\tilde{\gamma}\|, 1\}} + \theta^2\delta^2 \sum_{j:\, E^{\sharp\mu}_{j,L} \leq \mu} \frac{1}{|I_j(\delta)|} .$$
- $\|\tilde{\gamma}\| \leq 1 + 2|\theta|\delta L N_\mu$.

. . . finish with a Borel-Cantelli argument.

IV. Concluding remarks

1. No BEC and no superfluidity in the disordered TG gas
2. In other parameter regimes different behavior is possible:
   - BEC and superfluidity at $T = 0$ with mean-field-type interaction Könenberg/Moser/Seiringer/Yngvason '15
   - Numerical phase-diagram for disordered Bose-Hubbard model at incommensurate filling Rapsch/Schollwöck/Zwerger '99



3. Tonks-Girardeau gas is the continuum analogue of XY spin chain:
   - reduction of the many-particle problem to one-particle properties
   - difficulty: many-particle physical correlations reduce to **multi-point correlation functions** of the underlying one-particle problem **new estimates** Sims/W. '15

THANK YOU!