Small eigenvalues for some non local semiclassical linear Boltzmann equations

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The equations

Linear relaxation Boltzmann equation

$$\begin{cases} h\partial_t u + v.h\partial_x u - \partial_x V(x).h\partial_v u + h(\mathrm{Id} - \Pi_h)u = 0, \\ u_{|t=0} = u_0, \end{cases}$$

where Π_h is the orthogonal projector on the space of local equilibria $E_h = \left\{ \rho\mu_h^{\frac{1}{2}},\ \rho \in L^2(\mathbb{R}_x^d) \right\}$ with $\mu_h = \mathrm{e}^{-\frac{v^2}{2h}}$.

The equations

Soft relaxation linear Boltzmann equation

$$\begin{cases} h\partial_t u + v.h\partial_x u - \partial_x V(x).h\partial_v u + (1+H_0)^{-1} H_0 u = 0, \\ u_{|t=0} = u_0, \end{cases}$$

where H_0 is the semiclassical harmonic oscillator in velocity (on $\mathbb{R}^d_v$), i.e.

$$H_0 = -h^2\Delta_v + \frac{v^2}{4} - \frac{hd}{2}.$$

Hypotheses

- The potential $V \in \mathcal{C}^\infty(\mathbb{R}^d_x, \mathbb{R})$ has n_0 local (non-degenerate) minima ;
- The derivative of V of order 2 or more are bounded ;
- the potential is confining: $\mathrm{e}^{-\frac{V}{h}} \in L^2$ and there exists $C > 0$ such that $|\nabla V(x)| \geq \dfrac{1}{C}$ pour $|x| > C$.

Main theorem

Theorem

0 is a simple eigenvalue of P_h and there exists $h_0 > 0$, $\delta_0 > 0$ such that:

i) for all $h \in]0, h_0]$, $\mathrm{Spec}\, P_h \cap B(0, \delta_0 h)$ consists in n_0 (counted for multiplicity) real eigenvalues which are all exponentially small with respect to $\frac{1}{h}$;

ii) for all $\delta > 0$, there exists $C > 0$ such that for all $h \in]0, h_0]$, if $\delta h \leq |z| \leq \delta_0 h$ then

$$\|(P_h - z)^{-1}\| \leq \frac{C}{h}.$$

Supersymmetry

Proposition

The operator $P = v.h\partial_x - \partial_x V(x).h\partial_v + (1+H_0)^{-1}H_0$ coincides on functions with the following supersymmetric operator (defined on the set of k-forms):

$$\mathcal{P} = \tilde{\mathrm{d}}^{A,*}\tilde{\mathrm{d}} + \tilde{\mathrm{d}}\tilde{\mathrm{d}}^{A,*},$$

where $\tilde{\mathrm{d}} = (1+2h+H)^{\frac{1}{2}}a + (1+H)^{-\frac{1}{2}}b$, with H the semiclassical harmonic oscillator in velocity on k-forms.

Witten complex

We introduce the twisted Witten (de Rham) complex for $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$:

$$\mathrm{d}_\varphi = \mathrm{e}^{-\varphi/h} \circ h\mathrm{d} \circ \mathrm{e}^{\varphi/h} = h\mathrm{d} + (\mathrm{d}\varphi)^\wedge,$$

where d is the usual exterior derivative.

Auxiliary operators

We put

$$a = (h\mathrm{d} + (\mathrm{d}V)^{\wedge}), \qquad b = (h\mathrm{d} + (\mathrm{d}\varphi)^{\wedge}) \ ,$$

where a and b respectively act on $\mathcal{C}_0^\infty(\mathbb{R}_x^d; \Lambda^k T^*\mathbb{R}_x^d)$ and $\mathcal{C}_0^\infty(\mathbb{R}_v^d; \Lambda^k T^*\mathbb{R}_v^d)$ with $\varphi(v) = v^2/2$.

Auxiliary operators

We also consider the semiclassical harmonic oscillator

$$H = b^*b + bb^* = H_0 \otimes \mathrm{Id} + h\sum_{j=1}^{d} \mathrm{d}v_j^\wedge \left(\frac{\partial}{\partial v_j}\right)^\rfloor,$$

where $H_0 = -h^2\Delta + v^2 - hd$.
We also introduce the auxiliary operator:

$$\Lambda^2_{\alpha(h)} = \alpha(h) + H, \qquad \alpha(h) \geq \alpha > 0.$$

The "exterior derivative"

We put

$$\tilde{\mathrm{d}} = \Lambda_{1+2h} a + \Lambda_1^{-1} b.$$

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We notice that $\tilde{\mathrm{d}} : \Lambda^k T^*\mathbb{R}^{2d} \to \Lambda^{k+1} T^*\mathbb{R}^{2d}$ verifies:

$$\tilde{\mathrm{d}}^2 = 0.$$

Bilinear form

$$A\,:\,T^*\mathbb{R}^n \to T\mathbb{R}^n,$$

an invertible linear application. We then define the non degenerate bilinear form induced by A on k-forms:

$$(u|v)_A = v\left(\Lambda^k A(u)\right), \quad u,\; v \in \Lambda^k T^*\mathbb{R}^n.$$

If $a\,:\,\Lambda^k T^*\mathbb{R}^n \to \Lambda^l T^*\mathbb{R}^n$ is a linear application, we define the "adjoint" $a^{A,*}\,:\,\Lambda^l T^*\mathbb{R}^n \to \Lambda^k T^*\mathbb{R}^n$ by

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In our case we take

$$A = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

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Semiclassical Weyl symbol

$$
\begin{aligned}
p &= \frac{v^2+\eta^2-hd}{1+v^2+\eta^2-hd} + i(v.\xi - V'(x).\eta) + \mathcal{O}(h^2) \\
&= \frac{v^2+\eta^2}{1+v^2+\eta^2} + i(v.\xi - V'(x).\eta) - \frac{hd}{(1+v^2+\eta^2)^2} + \mathcal{O}(h^2) \\
&= p_0 + ip_1 + hp_2 + \mathcal{O}(h^2).
\end{aligned}
$$

Away from critical points

We put for $\alpha > 0$

$$g = -\frac{\sqrt{\alpha} v.V'(x) + \sqrt{\alpha}\eta.\xi}{2(1 + |V'(x)|^2 + \alpha v^2 + \xi^2 + \alpha\eta^2)}.$$

Away from critical points

We want to bound from below

$$\mathrm{Re}\,((p - hz)^w u, (1 - g)^w u).$$

Which we rewrite

$$\mathrm{Re}\,(((p - hz)\#(1 - g))^w u, u).$$

Away from critical points

Thanks to symbolic calculus, we obtain:

$$\begin{aligned}\mathrm{Re}\,((p - hz)\#(1-g)) &= (p_0 - \mathrm{Re}\,hz)(1-g) + h\{p_1, g\}/2 \\ &\quad + hp_2 + O(h^2) \\ &\geq p_0 - C(\alpha)hp_0 \\ &\quad + \frac{h\sqrt{\alpha}}{4}\,\frac{\alpha v^2 + \alpha\eta^2 + V'(x)^2 + \xi^2}{1 + V'(x)^2 + \xi^2 + \alpha v^2 + \alpha\eta^2} \\ &\quad - (d + |\mathrm{Re}\,z|)h + O(h^2).\end{aligned}$$

Away from critical points

We fix a cut-off function $\varphi \in \mathcal{C}_0^\infty(T^*\mathbb{R}^{2d})$ taking value 1 near critical points. Outside the support of φ, we have

$$\frac{\alpha v^2 + \alpha\eta^2 + V'(x)^2 + \xi^2}{1 + V'(x)^2 + \xi^2 + \alpha v^2 + \alpha\eta^2} \geq c.$$

So taking α large enough (α is fixed from now on) we get

$$\begin{aligned}\mathrm{Re}\,((p - hz)\#(1-g)) \geq & p_0 - C(\alpha)hp_0 + c(1-\varphi)^2 h \\ & + O(h^2).\end{aligned}$$

We now take h small enough, we have

$$\mathrm{Re}\,((p - hz)\#(1-g)) \geq c(1-\varphi)^2 h.$$

Away from critical points

Thanks to Fefferman-Phong inequality, we get

$$\mathrm{Re}\,(((p-hz)\#(1-g))^w u, u) \geq ch\,\|(1-\varphi)^w u\|^2 + O(h^2)\|u\|^2.$$

Which we rewrite

$$\mathrm{Re}\,((p-hz)^w u, (1-g)^w u) \geq ch\,\|(1-\varphi)^w u\|^2 + O(h^2)\|u\|^2.$$

And finally

$$\|(P-hz)u\|\,\|u\| \geq ch\,\|(1-\varphi)^w u\|^2 + O(h^2)\|u\|^2.$$

Near critical points

We start from the estimate (on the FBI side) (6.14) from Hérau-Sjöstrand-Stolk:

$$\|\Lambda u\|^2 \leq C\,\mathrm{Re}\,(\chi(P-hz)u,u) + C\left(\chi_0\left(\frac{X}{\sqrt{Ah}}\right)\Lambda u, \Lambda u\right) + C\,\|(1-\chi)\Lambda u\|\,\|\Lambda u\|\,,$$

where $\Lambda = h + \min(\delta^2, (Ah\delta)^{2/3})$ and χ, $\chi_0 \in \mathcal{C}_0^\infty$ with support larger than φ.

Near critical points

We apply this inequality to $\varphi^w u$ and we get

$$\begin{aligned}\|\Lambda\varphi^w u\|^2 \leq & \, C\,\mathrm{Re}\left(\chi(P-hz)\varphi^w u, \varphi^w u\right) \\ & + C\left(\chi_0\left(\frac{X}{\sqrt{Ah}}\right)\Lambda\varphi^w u, \Lambda\varphi^w u\right) \\ & + C\left\|(1-\chi)\Lambda\varphi^w u\right\|\left\|\Lambda\varphi^w u\right\|.\end{aligned}$$

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$$\|(1-\chi)\Lambda\varphi^w u\|\,\|\Lambda\varphi^w u\| = O(h^\infty)\,\|u\|^2.$$

Near critical points

For the first term, we first get

$$\mathrm{Re}\,(\chi(P-hz)\varphi^w u, \varphi^w u) = \mathrm{Re}\,((P-hz)\varphi^w u, \varphi^w u) + O(h^\infty)\,\|u\|^2\,.$$

Then with Fefferman-Phong inequality

$$\mathrm{Re}\,((P-hz)\varphi^w u, \varphi^w u) \leq \mathrm{Re}\,((P-hz)u, u) + O(h^2)\,\|u\|^2\,.$$

Quadratic approximation

For the remaining term, we use the following estimates on the quadratic approximation of P for $\delta \leq |z| \leq \delta_0$ (δ is fixed, but arbitrarily small) and $\mathrm{supp}\chi_0 \subset K$:

$$\begin{aligned}\left\|\Lambda\chi_0\left(\frac{x}{\sqrt{Ah}}\right)u\right\| \leq & C\left\|\Lambda^{-1}\chi_0\left(\frac{x}{\sqrt{Ah}}\right)(P_0 - hz)u\right\| \\ & + \frac{C}{\sqrt{A}}\left\|\Lambda 1_K\left(\frac{x}{\sqrt{Ah}}\right)u\right\|.\end{aligned}$$

$$\left\|\Lambda^{-1}\chi_0\left(\frac{x}{\sqrt{Ah}}\right)(P - P_0)u\right\| \leq C(A)h^{\frac{1}{2}}\left\|\Lambda u\right\|.$$

Near critical points

Combining all those estimate, we get

$$h^2 \|\varphi^w u\|^2 \leq C(B) \|(P - hz)u\|^2 + \frac{h^2}{B} \|u\|^2 + \mathcal{O}(h^3) \|u\|^2 .$$

with B that can be chosen arbitrarily large.

Near critical points

Combining all those estimate, we get

$$h^2 \left\|\varphi^w u\right\|^2 \leq C(B) \left\|(P - hz)u\right\|^2 + \frac{h^2}{B} \left\|u\right\|^2 + \mathcal{O}(h^3) \left\|u\right\|^2 .$$

with B that can be chosen arbitrarily large.
We had

$$\|(P - hz)u\| \, \|u\| \geq ch \left\|(1 - \varphi)^w u\right\|^2 + O(h^2)\|u\|^2.$$

We put this estimates together and get the wanted resolvent estimate.

Thanks for your attention