Block-modified random matrices, operator-valued free probability, and applications to entanglement theory

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Entanglement in Quantum Information Theory

- Quantum states with n degrees of freedom are described by density matrices
$$\rho \in \mathbb{M}_n^{1,+} = \mathrm{End}^{1,+}(\mathbb{C}^n); \qquad \mathrm{Tr}\rho = 1 \text{ and } \rho \geq 0$$
- Two quantum systems: $\rho_{12} \in \mathrm{End}^{1,+}(\mathbb{C}^m \otimes \mathbb{C}^n) = \mathbb{M}_{mn}^{1,+}$
- A state ρ_{12} is called separable if it can be written as a convex combination of product states
$$\rho_{12} \in \mathcal{SEP} \iff \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i),$$
where $t_i \geq 0$, $\sum_i t_i = 1$, $\rho_1(i) \in \mathbb{M}_m^{1,+}$, $\rho_2(i) \in \mathbb{M}_n^{1,+}$
- Equivalently, $\mathcal{SEP} = \mathrm{conv}\left[\mathbb{M}_m^{1,+} \otimes \mathbb{M}_n^{1,+}\right]$
- Non-separable states are called entangled

More on entanglement - pure states

- Separable rank one (pure) states $\rho_{12} = P_{e \otimes f} = P_e \otimes P_f$.
- Bell state or maximally entangled state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2) \neq x \otimes y.$$

- For rank one quantum states, entanglement can be detected and quantified by the entropy of entanglement

$$E_{\mathrm{ent}}(P_x) = H(s(x)) = -\sum_{i=1}^{\min(m,n)} s_i(x) \log s_i(x),$$

  where $x \in \mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{M}_{m \times n}(\mathbb{C})$ is seen as a $m \times n$ matrix and $s_i(x)$ are its singular values.
- A pure state $x \in \mathbb{C}^m \otimes \mathbb{C}^n$ is separable $\iff$ $E_{\mathrm{ent}}(P_x) = 0$.

Separability criteria

- Let $\mathcal{A}$ be a C^* algebra. A map $f : \mathbb{M}_n \to \mathcal{A}$ is called
  - positive if $A \geq 0 \implies f(A) \geq 0$;
  - completely positive (CP) if $\mathrm{id}_k \otimes f$ is positive for all $k \geq 1$ ($k = n$ is enough).
- Let $f : \mathbb{M}_n \to \mathcal{A}$ be a completely positive map. Then, for every state $\rho_{12} \in \mathbb{M}_{mn}^{1,+}$, one has $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$.
- Let $f : \mathbb{M}_n \to \mathcal{A}$ be a positive map. Then, for every separable state $\rho_{12} \in \mathbb{M}_{mn}^{1,+}$, one has $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$.
  - ρ_{12} separable $\implies \rho_{12} = \sum_i t_i \rho_1(i) \otimes \rho_2(i)$.
  - $[\mathrm{id}_m \otimes f](\rho_{12}) = \sum_i t_i \rho_1(i) \otimes f(\rho_2(i))$.
  - For all i, $([\rho_2(i)) \geq 0$, so $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$.
- Hence, positive, but not CP maps f provide sufficient entanglement criteria: if $[\mathrm{id}_m \otimes f](\rho_{12}) \not\geq 0$, then ρ_{12} is entangled.
- Moreover, if $[\mathrm{id}_m \otimes f](\rho_{12}) \geq 0$ for all positive, but not CP maps $f : \mathbb{M}_n \to \mathbb{M}_m$, then ρ_{12} is separable.
- Actually, for the exact converse to hold, uncountably many positive maps are needed [Skowronek], and for a very rough approximation of $\mathcal{SEP}$, exponentially many positive maps are needed [Aubrun, Szarek].

Positive Partial Transpose matrices

- The transposition map $\mathrm{t} : A \mapsto A^t$ is positive, but not CP. Define the convex set
$$\mathcal{PPT} = \{\rho_{12} \in \mathbb{M}_{mn}^{1,+} \mid [\mathrm{id}_m \otimes \mathrm{t}_n](\rho_{12}) \geq 0\}.$$
- For $(m, n) \in \{(2, 2), (2, 3)\}$ we have $\mathcal{SEP} = \mathcal{PPT}$. In other dimensions, the inclusion $\mathcal{SEP} \subset \mathcal{PPT}$ is strict.
- Low dimensions are special because every positive map $f : \mathbb{M}_2 \to \mathbb{M}_{2/3}$ is decomposable:
$$f = g_1 + g_2 \circ \mathrm{t},$$
with $g_{1,2}$ completely positive. Among all decomposable maps, the transposition criterion is the strongest.

The PPT criterion at work

- Recall the Bell state $\rho_{12} = P_{\mathrm{Bell}}$, where

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \ni \mathrm{Bell} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2).$$

- Written as a matrix in $\mathbb{M}^{1,+}_{2\cdot 2}$

$$\rho_{12} = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) = \frac{1}{2}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

- Partial transposition: transpose each block B_{ij}:

$$\rho_{12}^{\Gamma} = [\mathrm{id}_2 \otimes \mathrm{t}_2](\rho_{12}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- This matrix is no longer positive $\Longrightarrow$ the state is entangled.

The Choi matrix of a map

- For any n, recall that the maximally entangled state is the orthogonal projection onto

$$\mathbb{C}^n \otimes \mathbb{C}^n \ni \mathrm{Bell} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \otimes e_i.$$

- To any map $f : \mathbb{M}_n \to \mathcal{A}$, associate its Choi matrix

$$C_f = [\mathrm{id}_n \otimes f](P_{\mathrm{Bell}}) \in \mathbb{M}_n \otimes \mathcal{A}.$$

- Equivalently, if E_{ij} are the matrix units in $\mathbb{M}_n$, then

$$C_f = \sum_{i,j=1}^{n} E_{ij} \otimes f(E_{ij}).$$

Theorem (Choi '72)

A map $f : \mathbb{M}_n \to \mathcal{A}$ is CP iff its Choi matrix C_f is positive.

The Choi-Jamiołkowski isomorphism

- Recall (from now on $\mathcal{A} = \mathbb{M}_d$)

$$C_f = [\mathrm{id}_n \otimes f](P_{\mathrm{Bell}}) = \sum_{i,j=1}^{n} E_{ij} \otimes f(E_{ij}) \in \mathbb{M}_n \otimes \mathbb{M}_d.$$

- The map $f \mapsto C_f$ is called the Choi-Jamiołkowski isomorphism.
- It sends:
  1. All linear maps to all operators;
  2. Hermicity preserving maps to hermitian operators;
  3. Entanglement breaking maps to separable quantum states;
  4. Unital maps to operators with unit left partial trace ($[\mathrm{Tr} \otimes \mathrm{id}]C_f = \mathrm{I}_d$);
  5. Trace preserving maps to operators with unit left partial trace ($[\mathrm{id} \otimes \mathrm{Tr}]C_f = \mathrm{I}_n$).

How powerful are the entanglement criteria?

- Let $f : \mathbb{M}_m \to \mathbb{M}_n$ be a given linear map (f positive, but not CP).
- If $[f \otimes \mathrm{id}](\rho) \not\geq 0$, then $\rho \in \mathbb{M}_m \otimes \mathbb{M}_d$ is entangled.
- If $[f \otimes \mathrm{id}](\rho) \geq 0$, then … we do not know.
- Define

$$\mathcal{K}_f := \{\rho \, : \, [f \otimes \mathrm{id}](\rho) \geq 0\} \supseteq \mathcal{SEP}.$$

- We would like to compare (e.g. using the volume) the sets $\mathcal{K}_f$ and $\mathcal{SEP}$.
- Several probability measures on the set $\mathbb{M}_{md}^{1,+}$: for any parameter $s \geq md$, let W be a Wishart matrix of parameters (md, s):

$$W = XX^*, \text{ with } X \in \mathbb{M}_{md \times s} \text{ a Ginibre random matrix.}$$

- Let $\mathbb{P}_s$ be the probability measure obtained by pushing forward the Wishart measure by the map $W \mapsto W/\mathrm{Tr}(W)$.
- To compute $\mathbb{P}_s(\mathcal{K}_f)$, one needs to decide whether the spectrum of the random matrix $[f \otimes \mathrm{id}](W)$ is positive (here, d is large, m, n are fixed) $\rightsquigarrow$ block modified matrices.

Block-modified random matrices - previous results

Many cases studied independently, using the method of moments; no unified approach, each case requires a separate analysis:

- [Aubrun '12]: the asymptotic spectrum of $W^\Gamma := [\mathrm{id} \otimes \mathrm{t}](W)$ is a shifted semicircular, for $W \in \mathbb{M}_d \otimes \mathbb{M}_d$, $d \to \infty$
- [Banica, N. '13]: the asymptotic spectrum of $W^\Gamma := [\mathrm{id} \otimes \mathrm{t}](W)$ is a free difference of free Poisson distributions, for $W \in \mathbb{M}_m \otimes \mathbb{M}_d$, $d \to \infty$, m fixed
- [Jivulescu, Lupa, N. '14,'15]: the asymptotic spectrum of $W^{red} := W - [\mathrm{Tr} \otimes \mathrm{id}](W) \otimes I$ is a compound free Poisson distribution, for $W \in \mathbb{M}_m \otimes \mathbb{M}_d$, $d \to \infty$, m fixed (here, $f(X) = X - \mathrm{Tr}(X) \cdot I$)
- etc...

$\rightsquigarrow$ we propose a general, unified framework for such problems

The problem

- Consider a sequence of unitarily invariant random matrices $X_d \in \mathbb{M}_n \otimes \mathbb{M}_d$, having limiting spectral distribution μ.
- Define the modified version of X_d:

$$X_d^f = [f \otimes \mathrm{id}_d](X_d).$$

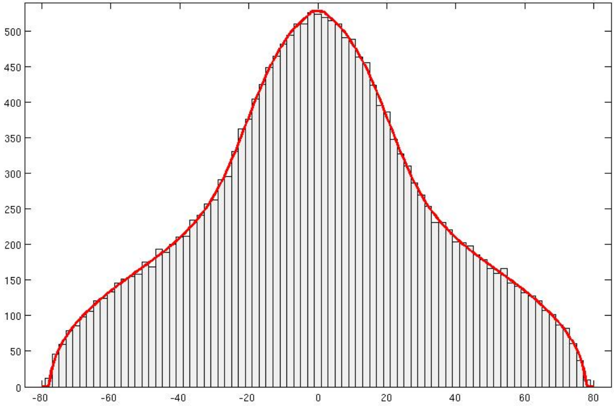
- Our goal: compute μ^f, the limiting spectral distribution of $\hat{X}_d$, as a function of
    1. The initial distribution μ
    2. The function f.

- Results: achieved this for all μ and a fairly large class of f.
- Tools: operator-valued free probability theory.

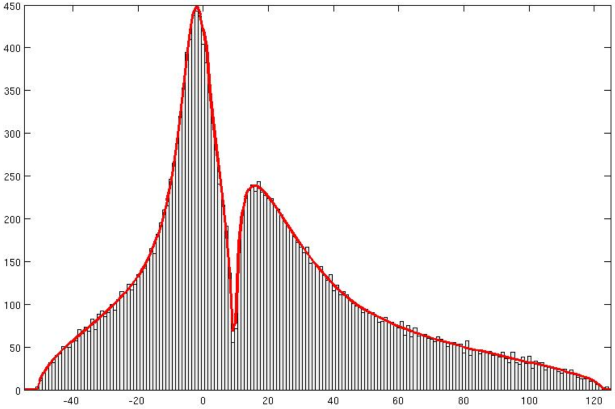
An example

$$f\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 11a_{11} + 15a_{22} - 25a_{12} - 25a_{21} & 36a_{21} \\ 36a_{12} & 11a_{11} - 4a_{22} \end{bmatrix}$$

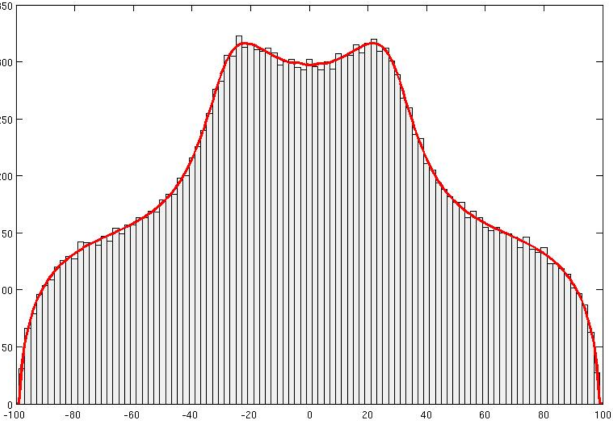
Wigner distribution



Wishart distribution



Arcsine distribution



Taking the limit

- We can write
$$X_d^f = [f \otimes \mathrm{id}](X_d) = \sum_{i,j,k,l=1}^{n} c_{ijkl}(E_{ij} \otimes I_d) X_d (E_{kl} \otimes I_d) \in \mathbb{M}_n \otimes \mathbb{M}_d,$$
for some coefficients $c_{ijkl} \in \mathbb{C}$, which are actually the entries of the Choi matrix of f.
- At the limit:
$$x^f = \sum_{i,j,k,l=1}^{n} c_{ijkl} e_{i,j} x e_{k,l},$$
for some random variable x having the same distribution as the limit of X_d and some (abstract) matrix units e_{ij}.

$\rightsquigarrow$ In the rectangular case $m \neq n$, one needs to use the techniques of Benaych-Georges; we will have freeness with amalgamation on $\langle p_m, p_m \rangle$.

Operator valued freeness

Definition

(1) Let $\mathcal{A}$ be a unital $*$-algebra and let $\mathbb{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ be a $*$-subalgebra. A $\mathcal{B}$-probability space is a pair $(\mathcal{A}, \mathbb{E})$, where $\mathbb{E} : \mathcal{A} \to \mathcal{B}$ is a conditional expectation, that is, a linear map satisfying:

$$\begin{aligned} \mathbb{E}\left(bab'\right) &= b\mathbb{E}(a)b', \qquad \forall b, b' \in \mathcal{B}, a \in \mathcal{A} \\ \mathbb{E}\left(1\right) &= 1. \end{aligned}$$

(2) Let $(\mathcal{A}, \mathbb{E})$ be a $\mathcal{B}$-probability space and let $\bar{a} := a - \mathbb{E}(a)1_{\mathcal{A}}$ for any $a \in \mathcal{A}$. The $*$-subalgebras $\mathcal{B} \subseteq A_1, \ldots, A_k \subseteq \mathcal{A}$ are $\mathcal{B}$-free (or free over $\mathcal{B}$, or free with amalgamation over $\mathcal{B}$) (with respect to $\mathbb{E}$) iff

$$\mathbb{E}(\bar{a}_1 \bar{a}_2 \cdots \bar{a}_r) = 0,$$

for all $r \geq 1$ and all tuples $a_1, \ldots, a_r \in \mathcal{A}$ such that $a_i \in A_{j(i)}$ with $j(1) \neq j(2) \neq \cdots \neq j(r)$.
(3) Subsets $S_1, \ldots, S_k \subset \mathcal{A}$ are $\mathcal{B}$-free if so are the $*$-subalgebras $\langle S_1, \mathcal{B}\rangle, \ldots, \langle S_k, \mathcal{B}\rangle$.

Similar to independence, freeness allows to compute mixed moments free random variables in terms of their individual moments.

Matrix-valued probability spaces

Let $\mathcal{A}$ be a unital C^*-algebra and let $\tau : \mathcal{A} \to \mathbb{C}$ be a state. Consider the algebra $\mathbb{M}_n(\mathcal{A}) \cong \mathbb{M}_n \otimes \mathcal{A}$ of $n \times n$ matrices with entries in $\mathcal{A}$. The maps

$$\mathbb{E}_3 : (a_{ij})_{ij} \mapsto (\tau(a_{ij}))_{ij} \in \mathbb{M}_n,$$

$$\mathbb{E}_2 : (a_{ij})_{ij} \mapsto (\delta_{ij}\tau(a_{ij}))_{ij} \in \mathbb{D}_n,$$

and

$$\mathbb{E}_1 : (a_{ij})_{ij} \mapsto \sum_{i=1}^{n} \frac{1}{n}\tau(a_{ii}) I_n \in \mathbb{C} \cdot I_n$$

are respectively, conditional expectations onto the algebras $\mathbb{M}_n \supset \mathbb{D}_n \supset \mathbb{C} \cdot I_n$ of constant matrices, diagonal matrices and multiples of the identity.

Proposition

If $A_1, \ldots, A_k$ are free in $(\mathcal{A}, \tau)$, then the algebras $M_n(A_1), \ldots, M_n(A_k)$ of matrices with entries in $A_1, \ldots, A_k$ respectively are in general not free over $\mathbb{C}$ (with respect to $\mathbb{E}_1$). They are, however, $\mathbb{M}_n$-free (with respect to $\mathbb{E}_3$).

Restricting cumulants

Proposition (Nica, Shlyakhtenko, Speicher)

Let $1 \in \mathcal{D} \subset \mathcal{B} \subset \mathcal{A}$ be algebras such that $(\mathcal{A}, \mathbb{F})$ and $(\mathcal{B}, \mathbb{E})$ are respectively $\mathcal{B}$-valued and $\mathcal{D}$-valued probability spaces and let $a_1, \ldots, a_k \in \mathcal{A}$. Assume that the $\mathcal{B}$-cumulants of $a_1, \ldots, a_k \in \mathcal{A}$ satisfy

$$R^{\mathcal{B};a_1,\ldots,a_k}_{i_1,\ldots,i_n}(d_1, \ldots, d_{n-1}) \in \mathcal{D},$$

for all $n \in \mathbb{N}$, $1 \leq i_1, \ldots, i_n \leq k$, $d_1, \ldots, d_{n-1} \in \mathcal{D}$.
Then the $\mathcal{D}$-cumulants of $a_1, \ldots, a_k$ are exactly the restrictions of the $\mathcal{B}$-cumulants of $a_1, \ldots, a_k$, namely for all $n \in \mathbb{N}$, $1 \leq i_1, \ldots, i_n \leq k$, $d_1, \ldots, d_{n-1} \in \mathcal{D}$:

$$R^{\mathcal{B};a_1,\ldots,a_k}_{i_1,\ldots,i_n}(d_1, \ldots, d_{n-1}) = R^{\mathcal{D};a_1,\ldots,a_k}_{i_1,\ldots,i_n}(d_1, \ldots, d_{n-1}),$$

Corollary

Let $\mathcal{B} \subseteq A_1, A_2 \subseteq \mathcal{A}$ be $\mathcal{B}$-free and let $\mathcal{D} \subseteq M_N(\mathbb{C}) \otimes \mathcal{B}$. Assume that, individually, the $\mathbb{M}_N \otimes \mathcal{B}$-valued moments (or, equivalently, the $\mathbb{M}_N \otimes \mathcal{B}$-cumulants) of both $x \in \mathbb{M}_N \otimes A_1$ and $y \in \mathbb{M}_N \otimes A_2$, when restricted to arguments in $\mathcal{D}$, remain in $\mathcal{D}$. Then x, y are $\mathcal{D}$-free.

A different formulation

Proposition

The block-modified random variable x^f has the following expression in terms of the eigenvalues and of the eigenvectors of the Choi matrix C:

$$x^f = v^*(x \otimes C)v,$$

where

$$v = \sum_{s=1}^{n^2} b_s^* \otimes a_s \in \mathcal{A} \otimes \mathbb{M}_{n^2},$$

a_s are the eigenvectors of C, and the random variables $b_s \in \mathcal{A}$ are defined by $b_s = \sum_{i,j=1}^n \langle E_i \otimes E_j, a_s \rangle e_{i,j}$.

Theorem

Consider a linear map $f : \mathbb{M}_n \to \mathbb{M}_n$ having a Choi matrix $C \in \mathbb{M}_{n^2} \subset \mathcal{A} \otimes \mathbb{M}_{n^2}$ which has tracially well behaved eigenspaces. Then, the random variables $x \otimes C$ and vv^* are free with amalgamation over the (commutative) unital algebra $\mathcal{B} = \langle C \rangle$ generated by the matrix C.

Well behaved functions

Definition

We say that f is well behaved if the eigenspaces of its Choi matrix are *tracially well behaved* if

$$\tau(b_{j_1} b_{j_2}^* Q_{i_1} \dots Q_{i_k}) = \delta_{j_1 j_2} \tau(b_{j_1} b_{j_1}^* Q_{i_1} \dots Q_{i_k}),$$

for every $i_1, \dots, i_k \leq n^2$ and $j_1, j_2 \leq n^2$. We define

$$Q_i = b_i^* b_i.$$

$\rightsquigarrow$ a stronger condition:

Definition

The Choi matrix C is said to satisfy the unitarity condition if, for all t, there is some real constant d_t such that $[\mathrm{id} \otimes \mathrm{Tr}](P_t) = d_t I_n$, where P_t are the eigenprojectors of C.

The limiting distributions of block-modified matrices

Theorem

If the Choi matrix C satisfies the unitarity condition, then the distribution of x^f has the following R-transform:

$$R_{x^f}(z) = \sum_{i=1}^{s} d_i \rho_i R_x \left[\frac{\rho_i}{n} z \right],$$

where ρ_i are the distinct eigenvalues of C and nd_i are ranks of the corresponding eigenprojectors. In other words, if μ, resp. μ^f, are the respective distributions of x and x^f, then

$$\mu^f = \boxplus_{i=1}^{s} (D_{\rho_i/n}\mu)^{\boxplus nd_i}.$$

Range of applications

The following functions are well behaved

1. Unitary conjugations $f(X) = UXU^*$
2. The trace and its dual $f(X) = \mathrm{Tr}(X)$, $f(x) = xI_n$
3. The transposition $f(X) = X^\top$
4. The reduction map $f(X) = I_n \cdot \mathrm{Tr}(X) - X$
5. Linear combinations of the above $f(X) = \alpha X + \beta \mathrm{Tr}(X) I_n + \gamma X^\top$
6. Mixtures of orthogonal automorphisms

$$f(X) = \sum_{i=1}^{n^2} \alpha_i U_i X U_i^*,$$

for orthogonal unitary operators U_i

$$\mathrm{Tr}(U_i U_j^*) = n\delta_{ij}.$$

The End

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