On the center of mass for asymptotically hyperbolic initial data sets

8ème rencontre du GDR *DynQua*

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Outline

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- Remark: $\int_{\mathbb{R}^3} \vec{x}\, \rho(\vec{x}) d\mu^\delta$ may not converge for $\rho \in L^1$.

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- Schwarzschild spacetime $(\mathcal{N}_m, \gamma_m)$ with mass $m > 0$:

$$\mathcal{N}_m = \mathbb{R} \times \left(\mathbb{R}^3 \setminus \{0, m/2\}\right),$$

$$\gamma_m = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 \delta,$$

where $\delta = dx^2 + dy^2 + dz^2$ is the Euclidean metric.

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- Spherical symmetry... but $(t, \vec{0}) \notin \mathcal{N}_m$.

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- γ solves Einstein equations $\Rightarrow$ (g, k) satisfies the *Constraints*

$$R(g) - 2\Lambda - |k|_g^2 + (\mathrm{tr}^g k)^2 = 16\pi T_{nn}$$
$$\nabla^i(k_{ij} - (\mathrm{tr}^g k)g_{ij}) = 8\pi T_{nj}.$$

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Choquet-Bruhat's Theorem (1952)

If (M^3, g, k) satisfies the Constraints, then there exists a spacetime $(\mathcal{N}^4, \gamma)$ solution of the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ in which (M^3, g, k) embeds as a spacelike hypersurface.

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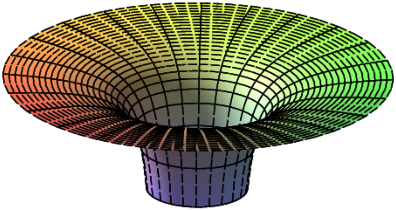

Figure : The end $M \setminus K$.

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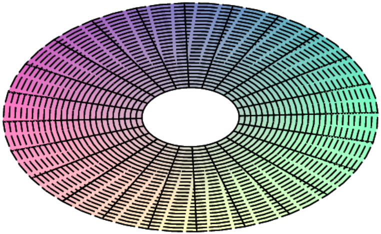

Figure : The image on $\mathbb{R}^3$.

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$$|\phi_* g - \delta| \leq Cr^{-1},\ |\nabla \phi_* g| \leq Cr^{-2},\ |\nabla^2 \phi_* g| \leq Cr^{-3},$$
$$|\phi_* k| \leq Cr^{-2},\ |\nabla \phi_* k| \leq Cr^{-3}.$$

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|\phi_* g - b| &\leq C(1-r)^3, \ldots, \\
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- Hyperbolic space $\mathbb{H}^3 = (B_1(0), b)$, with $b = \left(\frac{2}{1-r^2}\right)^2 \delta.$
- The reference spacetime is Anti-de Sitter spacetime, and $\Lambda = -3$.

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- Mass: $m(\phi, g) = \dfrac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} (\mathrm{div}\, e - d\, \mathrm{tr} e) \left(\dfrac{\vec{x}}{r}\right) dS_r$, where $e := \phi_* g - \delta$.

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- Center of mass: $\vec{c}(\phi, g) = (c^1, c^2, c^3) \in \mathbb{R}^3$, where

$$c^i = \frac{1}{16\pi m} \lim_{r\to\infty} \int_{S_r} \Big[x^i (\mathrm{div}\, e - d\, \mathrm{tr} e) - e(\nabla x^i, \cdot) + \mathrm{tr} e\, dx^i \Big] \left(\frac{\vec{x}}{r}\right) dS_r.$$

Foliations at infinity and CMC center for isolated systems

Theorem (Huisken-Yau, 1998)

If (M, g, k) asymptotically Euclidean with $m > 0$, there exists a unique foliation by constant mean curvature 2-spheres of

$$M \setminus K = \bigcup_{H \in (0, H_1)} \Sigma_H,$$

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- Advantage of this approach: the foliation $\{\Sigma_H\}_H$ is geometric, *i.e.* does not depend at all on ϕ !

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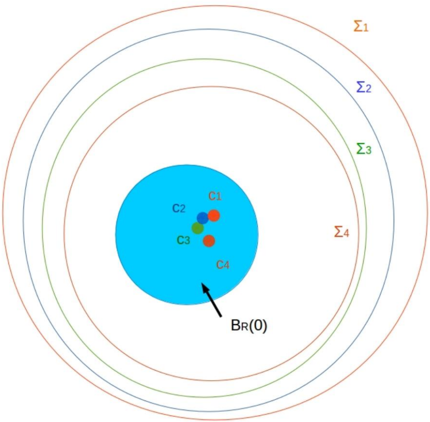

Figure : Hypersurfaces $\phi(\Sigma_H)$ and their affine centers in $\mathbb{R}^3$.

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For hyperbolic systems, the mass is a vector $\mathbf{p}(\phi, g) \in \mathbb{R}^{3,1}$. It is *positive* if it is timelike, future-directed: $p^0 > \sqrt{\sum_{i=1}^{3}(p^i)^2}$.

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Theorem (Neves-Tian, 2010)

If (M, g, k) asymptotically hyperbolic with positive mass vector, there exists a unique foliation by constant mean curvature 2-spheres of

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Theorem (Cederbaum, C., Sakovich 2015)

If (M, g, k) satisfies the assumptions of Neves-Tian's Theorem, we have the convergence in $\mathbb{H}^3$

$$\lim_{H \to 2} \mathbf{c}_H(\phi, g) = \frac{\mathbf{p}(\phi, g)}{|\mathbf{p}(\phi, g)|_{\mathbb{R}^{3,1}}}.$$

Thanks!