The Yang-Mills Fields on Black Hole Space-Times

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Context and motivation

The Einstein equations: $(M, \mathbf{g})$ 4-D Lorentzian manifold with $R_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}R = 8\pi T_{\mu\nu}$. In vacuum: $R_{\mu\nu} = 0$.

- Tensorial equations.
- Highly non-linear.

Exact solutions: The Minkowski space-time (flat), the Schwarzschild and the Kerr black holes, etc ...

One of the most famous results in General Relativity:

The stability of Minkowski space: by Christodoulou and Klainerman, 1993 (526 pages).

One of the biggest conjectures in General Relativity:

The stability of the Kerr black holes.

Toy model: $\Box_{\mathbf{g}}\phi = \nabla^{\alpha}\nabla_{\alpha}\phi = 0$.

- does not admit stationary solutions on the Schwarzschild black hole.
- Yang-Mills admits stationary solutions on Schwarzschild black hole.

Set up

- $(M, \mathbf{g})$: 4-D globally hyperbolic Lorentzian manifold.
- G : compact Lie group.
- $\mathcal{G}$: Lie algebra of G.
- $<\ ,\ >$: positive definite Ad-invariant scalar product on $\mathcal{G}$.
- The Yang-Mills potential: in a given system of coordinates, $\mathcal{G}$-valued one form A on M

$$A = A_\alpha dx^\alpha$$

The gauge covariant derivative of a $\mathcal{G}$-valued tensor Ψ is defined as:

$$\mathbf{D}^{(A)}_\alpha \Psi = \nabla_\alpha \Psi + [A_\alpha, \Psi]$$

- ∇_α : the space-time covariant derivative of Levi-Civita on $(M, \mathbf{g})$.
- $\nabla_\alpha \Psi$: the tensorial covariant derivative of an n-tensor Ψ:

$$(\nabla_Y \Psi)(f.X_1, \ldots, X_n) \quad = \quad f.(\nabla_Y \Psi)(X_1, \ldots, X_n)$$

- The tensorial second order derivative is defined such that:

$$(\nabla_Z \nabla_{f.Y} \Psi)(X_1, \ldots, X_n) \quad = \quad f.(\nabla_Z \nabla_Y \Psi)(X_1, \ldots, X_n)$$

- The Yang-Mills curvature is a $\mathcal{G}$-valued two form:

$$F = F_{\alpha\beta} dx^{\alpha} \wedge dx^{\beta}$$

obtained by commutating two gauge covariant derivatives of a $\mathcal{G}$-valued tensor $\Psi = \Psi_{a_1 a_2 \ldots . a_i \ldots .}$:

$$(\mathbf{D}^{(A)}_{\alpha} \mathbf{D}^{(A)}_{\beta} - \mathbf{D}^{(A)}_{\beta} \mathbf{D}^{(A)}_{\alpha}) \Psi a_1 \ldots a_i \ldots \quad = \quad \sum_i R_{a_i}{}^{\gamma}{}_{\alpha\beta} \Psi_{\ldots.\gamma \ldots.} + [F_{\alpha\beta}, \Psi]$$

$$\rightarrow F_{\alpha\beta} = \nabla_{\alpha} A_{\beta} - \nabla_{\beta} A_{\alpha} + [A_{\alpha}, A_{\beta}]$$

The Yang-Mills equations

The Yang-Mills Lagrangian is given by

$$L = -\frac{1}{4} < F_{\alpha\beta}, F^{\alpha\beta} >$$

The action principle gives

$$\mathbf{D}^{(A)}_{\alpha} F^{\alpha\beta} = 0 \tag{1}$$

On the other hand, we have the Bianchi identities:

$$\mathbf{D}^{(A)}_{\alpha} F_{\mu\nu} + \mathbf{D}^{(A)}_{\mu} F_{\nu\alpha} + \mathbf{D}^{(A)}_{\nu} F_{\alpha\mu} = 0 \tag{2}$$

The equations (1) and (2) form the Yang-Mills equations.

Maxwell equations

The Maxwell equations correspond to the abelian case where $[\ ,\ ] = 0$, and therefore $\mathbf{D}^{(A)} = \nabla$.

Einstein Equations in a Yang-Mills form

At a point p of the space-time, one can choose a normal frame, which means a frame such that $\mathbf{g}(e_\alpha, e_\beta)(p) = \text{diag}(-1, 1, \ldots, 1)$, and $\frac{\partial}{\partial \sigma}\mathbf{g}(e_\alpha, e_\beta)(p) = 0$. Cartan formalism consists in defining the connection 1-form at p,

$$(A)_{\alpha\beta}(X) = \mathbf{g}(\nabla_X e_\beta, e_\alpha)$$

If we define the Lie bracket $[A_\mu, A_\nu]$ as

$$([A_\mu, A_\nu])_{\alpha\beta} = (A_\mu)_\alpha{}^\lambda (A_\nu)_{\lambda\beta} - (A_\nu)_\alpha{}^\lambda (A_\mu)_{\lambda\beta}$$

and,

$$(F_{\mu\nu})_{\alpha\beta} = \left(\nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]\right)_{\alpha\beta}$$

then, we have

$$(F_{\mu\nu})_{\alpha\beta} = R_{\alpha\beta\mu\nu}$$

Well known proposition:

$R_{\mu\nu} = 0 => (\mathbf{D}^{(A)\mu} F_{\mu\nu})_{\alpha\beta} = 0$

Example : Spherically symmetric SU(2) Yang-Mills fields

- $G = SU(2)$: real Lie group of 2x2 unitary matrices of determinant 1.
- $\mathcal{G}$: the Lie algebra associated to G is $su(2)$: the antihermitian traceless 2x2 matrices.

In the exterior of the Schwarzschild black hole, the metric can be written as:

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + \frac{1}{(1 - \frac{2m}{r})}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

Ansatz:

$$A(t,r) \quad = \quad W(t,r)\tau_1 d\theta + W(t,r)\sin(\theta)\tau_2 d\phi + \cos(\theta)\tau_3 d\phi$$

where τ_i, $i \in \{1,2,3\}$: matrices which form a real basis of $su(2) = \mathcal{G}$.

If we define,

$$r^* \quad = \quad r + 2m \log(r - 2m)$$

then, we have:

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})dr^{*2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

- The conserved Yang-Mills energy is given by:

$$E_F(t) = \int_{r=2m}^{\infty} \int_{\mathbf{S}^2} (|\partial_t W|^2 + (|\partial_{r^*} W|^2 + \frac{(1 - \frac{2m}{r})[W^2 - 1]^2}{2r^2}) dr^* d\sigma^2$$

- The Yang-Mills equations translate on W as:

$$\rightarrow \partial_t^2 W - \partial_{r^*}^2 W = \frac{(1 - \frac{2m}{r})}{r^2} W[1 - W^2]$$

The global existence of Yang-Mills fields on curved space-times

Relies on gauge covariant wave equation with source term:

$$\mathbf{D}^{(A)\alpha}\mathbf{D}^{(A)}_{\alpha}F_{\mu\nu} = R_{\mu\gamma}F^{\gamma}{}_{\nu} + R_{\nu\gamma}F_{\mu}{}^{\gamma} + 2R_{\gamma\mu\nu\alpha}F^{\gamma\alpha} + 2[F_{\mu}{}^{\alpha}, F_{\nu\alpha}]$$

- Eardley-Moncrief: fundamental solution for the wave equation + Crönstrum gauge $\rightarrow$ Initial data $\sim H^1_{loc}(F)$ on Minkowski.
- Chruściel-Shatah: Friedlander parametrix + Crönstrum gauge $\rightarrow$ $H^2_{loc}(F)$ on curved space-times.
- Klainerman-Rodnianski: Kirchoff Sobolev parametrix $\rightarrow$ gauge independent proof of non-blow up $\rightarrow$ $H^0_{loc}(F)$ on Minkowski.

Klainerman-Rodnianski parametrix + Grönwall inequalities to control the energies $E_F(t)$ and $E_{\mathbf{D}^{(A)}F}(t)$ $\rightarrow$ Grönwall type inequality on $|F|$ on a small time interval depending on the point p
$\rightarrow$ Gauge independent proof $\rightarrow$ $H^1_{loc}(F)$ on curved space-times.

The statement

Let $(M, \mathbf{g})$ be a curved 4-dimensional Lorentzian manifold. We assume that $\mathbf{g}$ is sufficiently smooth, and M is globally hyperbolic. We know by then that there exist a smooth vector field $\frac{\partial}{\partial t}$ such that M is foliated by Cauchy hypersurfaces Σ_t. Let $\hat{t}$ be a unit timelike vector field orthogonal to Σ_t. We assume there exists $C_{loc}(t) \in L^1_{loc}$, such that:

$$|\pi^{\mu\nu}(\hat{t})|_{L^\infty_{loc(\Sigma_t)}} = |\frac{1}{2}[\nabla^\mu \hat{t}^\nu + \nabla^\nu \hat{t}^\mu]|_{L^\infty_{loc(\Sigma_t)}} \leq C_{loc}(t) \quad (3)$$

Define:

$$E^{\hat{t}}_F(t = t_0) = \int_{\Sigma_{t=t_0}} \frac{1}{2}|F|^2.dV_\Sigma$$

$$E^{\hat{t}}_{\mathbf{D}^{(A)}F}(t = t_0) = \int_{\Sigma_{t=t_0}} \frac{1}{2}|\mathbf{D}^{(A)}F|^2.dV_\Sigma$$

The theorem

Theorem

If $E^{\hat{t}}_{F}(t = t_0) < \infty$ *and* $E^{\hat{t}}_{\mathbf{D}^{(A)}F}(t = t_0) < \infty$ *then, the norm of the Yang-Mills curvature* $F_{\mu\nu}$ *of local solutions defined for all* $t \in [t_0, T)$ *will not blow-up in time* t*, i.e. at each point* q *of the space-time at time* $t_q = T$*, we have*

$$\lim_{p \to q,\ t_p < T} |F|(p) < \infty \ , \tag{4}$$

also,

$$\lim_{t \to T,\ t < T} E^{\hat{t}}_{F}(t) < \infty \ , \tag{5}$$

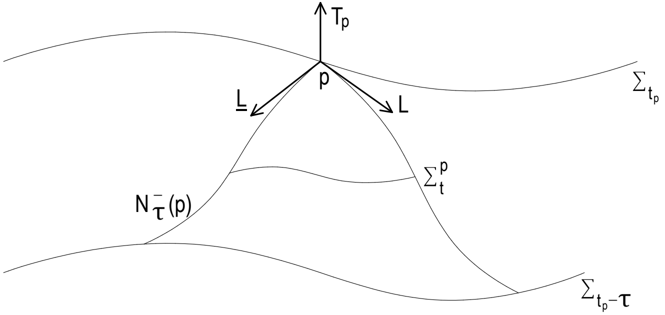
and

$$\lim_{t \to T,\ t < T} E^{\hat{t}}_{\mathbf{D}^{(A)}F}(t) < \infty \ . \tag{6}$$

The proof

Let $\mathbf{J}_p$ be a fixed $\mathcal{G}$-valued anti-symmetric 2-tensor at p, and let $\lambda_{\alpha\beta}$ be the unique 2-tensor field along $N^{-}(p)$, that verifies the linear transport equation:

$$\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}tr\chi\lambda_{\alpha\beta} = 0$$
$$(s\lambda_{\alpha\beta})(p) = \mathbf{J}_{\alpha\beta}(p)$$



An adaptation of the Kirchoff-Sobolev type representation formula of Klainerman-Rodnianski

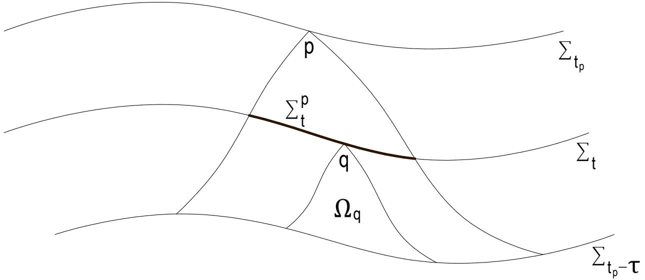
The scalar product is Ad-invariant: $< [A, B], C > = < A, [B, C] >$.
This yields to an adaptation of the parametrix to gauge covariant derivatives:

$$
\begin{aligned}
4\pi < \mathbf{J}_{\alpha\beta}, F^{\alpha\beta} > (p) &= -\int_{N^{-}_{\tau}(p)} < \lambda_{\alpha\beta}, \Box^{(A)}_{\mathbf{g}} F^{\alpha\beta} > + C_{t_p-\tau} \\
&+ \int_{N^{-}_{\tau}(p)} < \hat{\Delta}^{(A)} \lambda_{\alpha\beta} + 2\zeta_a \mathbf{D}^{(A)}_a \lambda_{\alpha\beta} + \frac{1}{2}\hat{\mu}\lambda_{\alpha\beta} \\
&+ [F_{L\underline{L}}, \lambda_{\alpha\beta}] - \frac{1}{2} R_{\alpha}{}^{\gamma}{}_{\underline{L}L} \lambda_{\gamma\beta} - \frac{1}{2} R_{\beta}{}^{\gamma}{}_{\underline{L}L} \lambda_{\alpha\gamma}, F^{\alpha\beta} >
\end{aligned}
$$

- where $C_{t_p-\tau}$ depends on the value of F on $\Sigma_{t_p-\tau}$.
- where $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ is the induced Laplacian on $\mathbb{S}^2$, of $\lambda_{\alpha\beta}$

The established Grönwall type inequality

$$||F||^2_{L^\infty(\Sigma^p_t)} \lesssim 1 + \int_{t_p-\tau}^{t} ||F||^2_{L^\infty(\Sigma^p_{\bar{t}})} d\bar{t} + \int_{t_p-\tau}^{t} \int_{t_p-\tau}^{t^*} ||F||^2_{L^\infty(\Sigma^p_{\bar{t}})} d\bar{t} dt^*$$



On the Yang-Mills fields on the Schwarzschild black hole

Results on black holes:

- Dafermos-Rodnianski $\rightarrow$ decay for Φ scalar solution of $\Box_g \Phi = 0$
- Blue $\rightarrow$ decay for Maxwell fields $\rightarrow$ separation of the middle components Φ_0 which satisfies $\Box_g \Phi_0 =$ linear term(Φ_0)
- Andersson-Blue $\rightarrow$ Kerr metrics.

Goal: get rid of the separation $\rightarrow$ not pass through the scalar wave equation.

Problem: getting a Morawetz type estimate.

- Stationary solutions are counter-examples for this estimate $->$ one needs to find a way to eliminate them in the proof.
- In the work of Blue, and Andersson-Blue, the problem is avoided by subtracting them, using the linearity of the Maxwell equations, and the fact that the only stationary solution is the Coulomb solution.

Spherically symmetric SU(2) Yang-Mills fields on the Schwarzschild black hole

$$\partial_t^2 W - \partial_{r^*}^2 W = \frac{(1-\frac{2m}{r})}{r^2} W[1-W^2]$$

- W_n : family of stationary solutions.
- $W_0 = \pm 1$ (zero Yang-Mills curvature) is stable. W_n, $n \neq 0$: unstable.

"Theorem" [G-Häfner]

There exists $\epsilon > 0$ such that

- if $E_F(t = t_0) < \epsilon$
- if some weighted energy is finite at $t = t_0$

$=>$then, the local energy decays in time t in the exterior of the black hole, i.e. for all r_1, r_2, such that $2m < r_1 \leq r_2$, we have at least

$$E_F(r_1 \leq r \leq r_2)(t) \leq \frac{C(r_1, r_2)}{t}$$

References

Sari Ghanem
The global non-blow up of Yang-Mills curvature on curved space-times
accepted for publication in Journal of Hyperbolic Differential Equations.

Sari Ghanem
On uniform decay of the Maxwell fields on black hole space-times
arXiv : 1409.8040, 114 pages.

Thank you!