Bubble of Nothing and Wormhole

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Outlines

Quantum Cosmology

- Universe : $U = (M, \mathcal{T}, \mathcal{A}, g; ...\Psi, \mathfrak{A}, \omega, ...)$
  Nothing $:= \emptyset$, *Minkowski, Kaluza-Klein, De Sitter,.....*
- Caveat : no choice of U is self-evident.
  Take seriously the category of all the universes : No obvious uniqueness of our universe. Existence of virtual universes.
  Wheeler, de Witt : Mini/Midi-superpace.
- Quantum tunneling (confusing, timeless) :

$$U_1 \rightsquigarrow U_2, \quad \emptyset \rightsquigarrow U, \quad U \rightsquigarrow \emptyset, ...$$

- *Witten 1985* : *Kaluza-Klein* $\rightsquigarrow$ *Bubble(s?) of Nothing*.
  Hawking 1987 : in quantum gravity, change of topology is possible ; loss of quantum coherence coming from a wormhole.
- Framework : Universe *ex nihilo* and *ad nihil* :

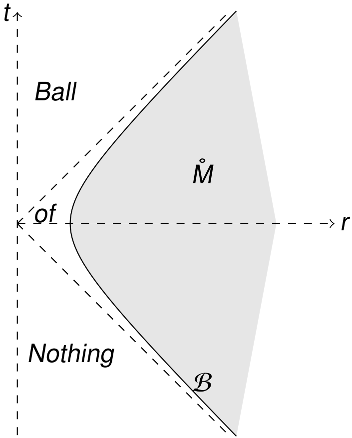
$$\emptyset \overset{t\to-\infty}{\longleftarrow} U \overset{t\to+\infty}{\longrightarrow} \emptyset$$

Issue : propagation of the classical and quantum waves.

Naive building

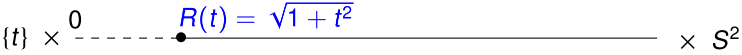
Take the space $\mathbf{R}^3_x$ and remove a ball of radius $R(t) = \sqrt{1+t^2}$

$$M = \bigcup_{t \in \mathbf{R}} \{t\} \times [R(t), \infty[_r \times S^2.$$



Trouble : time-like boundary $\mathcal{B}$ (Archytas argument).

Space $\{t\} \times [R(t), \infty[\times S^2$, *Bubble* $\{t\} \times \{R(t)\} \times S^2$



To avoid any boundary :

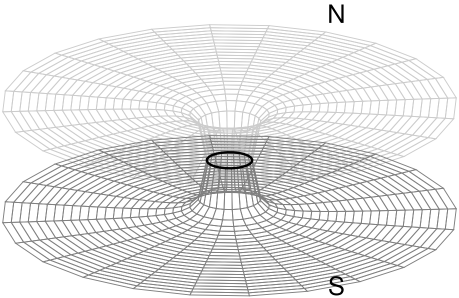
- to add two points $\{N, S\}$:

$$\{t\} \times [R(t), \infty[\times S^2 \times \{N, S\}, \quad \textit{identifying } N \textit{ and } S \textit{ at } R(t).$$

- to add a torus S^1 :

$$\{t\} \times [R(t), \infty[\times S^2 \times S^1, \quad \textit{identifying the points of } S^1 \textit{ at } R(t).$$

Topology



Equatorial Section : $\{t\} \times [R(t), \infty[\times \left(\{\theta = \pi/2\} \times S^1 \right) \times \{N, S\}$

Hawking Wormhole : $\mathbf{R}_t \times \mathbf{R} \times S^2$. Throat : $\mathbf{R}_t \times \{0_{\mathbf{R}}\} \times S^2$.
Witten Space-Time : $\mathbf{R}_t \times \mathbf{R}^2 \times S^2$. Bubble of Nothing : $\mathbf{R}_t \times \{0_{\mathbf{R}^2}\} \times S^2$.

Metrics

Witten method : double analytic continuation. Schwarzschild 5D :

$$ds^2 = \left(1 - \frac{R^2}{\rho^2}\right) dT^2 - \left(1 - \frac{R^2}{\rho^2}\right)^{-1} d\rho^2 - \rho^2 \left(d\Theta^2 + \sin^2 \Theta d\Omega_2^2\right), \quad \rho > R,$$

$$T = i\psi, \quad \psi \in S^1, \quad \Theta = \frac{\pi}{2} + it, \quad t \in \mathbf{R}$$

$$ds^2_{Witten} := \rho^2 dt^2 - \left(1 - \frac{R^2}{\rho^2}\right)^{-1} d\rho^2 - \rho^2 \cosh^2 t \; d\Omega_2^2 - \left(1 - \frac{R^2}{\rho^2}\right) d\psi^2$$

Einstein Vacuum Solution.

Submanifold $\{\psi = 0, \pi\}$ = Lorenzian Hawking Wormhole $\mathcal{W}$:

$$ds^2_{\mathcal{W}} = \rho^2 dt^2 - \left(1 - \frac{R^2}{\rho^2}\right)^{-1} d\rho^2 - \rho^2 \cosh^2 t \; d\Omega_2^2 \;\; (\text{on two sheets}).$$

$\{\rho = R\}$ = Minimal surface = Bubble of Nothing = Throat of the Wormhole = 3D-De Sitter space :

$$ds^2_{dS^3} := R^2 \left[dt^2 - \cosh^2 t \; d\Omega_2^2\right].$$

Witten space-time : coordinates ad nauseam

- Asymptotically Kaluza-Klein.
  Rindler : $\tau := \rho \sinh t$, $\xi := \rho \cosh t$, M : $\tau \in \mathbf{R}$, $\xi > \sqrt{\tau^2 + R^2}$

$$ds^2_{Witten} = d\tau^2 - d\xi^2 - \xi^2 d\Omega_2^2 - d\psi^2 + \frac{R^2}{\xi^2 - \tau^2}\left\{d\psi^2 - \frac{(\tau d\tau - \xi d\xi)^2}{\xi^2 - \tau^2 - R^2}\right\}$$

- Smoothness and Global Hyperbolicity ($R = 1$).

$$y := \frac{\sqrt{\rho^2 - 1}e^{\rho}}{1+\rho}\cos\psi, \quad z := \frac{\sqrt{\rho^2 - 1}e^{\rho}}{1+\rho}\sin\psi,$$

$$\rho = \frac{1}{2}W\left({}^{+2}_{-2}, y^2 + z^2\right) = 1 - 2\sum_{n=1}^{\infty}\frac{L_n'(4n)}{ne^{2n}}(y^2+z^2)^n$$

$$M = \mathbf{R}_t \times \mathbf{R}^2_{y,z} \times S^2.$$

$$ds^2_{Witten} = \rho^2 dt^2 - \frac{(1+\rho)^2}{\rho^2}e^{-2\rho}(dy^2 + dz^2) - \rho^2\cosh^2 t\; d\Omega_2^2,$$

Bubble of Nothing $\rho = 1, \quad y = z = 0, \quad ds^2_{dS^3} := dt^2 - \cosh^2 t\; d\Omega_2^2.$

$$\sqrt{\rho^2 - 1} = \sinh x, \quad M = \mathbf{R}_t \times [0, \infty[_x \times S^2 \times S^1$$

$$ds^2_{Witten} = \cosh^2 x \left[dt^2 - dx^2 - \cosh^2 t \, d\Omega_2^2 - \frac{\sinh^2 x}{\cosh^4 x} d\psi^2 \right]$$

$$T := \frac{\rho + \sqrt{\rho^2 - 1}}{2} \sinh t, \quad X := \frac{\rho + \sqrt{\rho^2 - 1}}{2} (\cosh t) \Omega_2 \in \mathbf{R}^3$$

$$M = \bigcup_{T \in \mathbf{T}} \{T\} \times \left\{ X \in \mathbf{R}^3, \mid X \mid^2 \geq T^2 + \frac{1}{4} \right\} \times S^1_\psi$$

$$ds^2_{Witten} = \left(1 + \frac{1}{4(\mid X \mid^2 - T^2)} \right)^2 \left\{ dT^2 - dX^2 - f(\mid X \mid^2 - T^2) d\psi^2 \right\}.$$

$$f(s) = 16 s^2 \frac{[4s - 1]^2}{[4s + 1]^4}$$

$$\textit{Bubble of Nothing} : \; x = 0, \quad \mid X \mid^2 = T^2 + \frac{1}{4}$$

Hawking Wormhole

$\mathcal{W}$: *submanifold* $\{\psi = 0, \pi\}$, *or* $\{z = 0\}$ *of* M

$$\mathcal{W} = \mathbf{R}_t \times \mathbf{R}_y \times S^2$$

$$ds^2_{\mathcal{W}} = \rho^2 dt^2 - \frac{(1+\rho)^2}{\rho^2} e^{-2\rho} dy^2 - \rho^2 \cosh^2 t d\Omega^2_2, \quad \rho = \frac{1}{2} W\left({}^{+2}_{-2}, y^2\right)$$

$$\mathcal{W} = \mathbf{R}_t \times \mathbf{R}_x \times S^2$$

$$ds^2_{\mathcal{W}} = \cosh^2(x) \left[dt^2 - dx^2 - \cosh^2 t d\Omega^2_2\right].$$

$$\mathcal{W} = \bigcup_{T \in \mathbf{R}} \left\{(T, X), \mid X \mid^2 \geq T^2 + \frac{1}{4}\right\}^2 / \left\{(T, X), \mid X \mid^2 = T^2 + \frac{1}{4}\right\}$$

$$ds^2_{\mathcal{W}} = \left(1 + \frac{1}{4(\mid X \mid^2 - T^2)}\right)^2 \left\{dT^2 - dX^2\right\}$$

Throat=Bubble of Nothing : $y = 0$, $x = 0$, $\mid X \mid^2 = T^2 + \frac{1}{4}$.

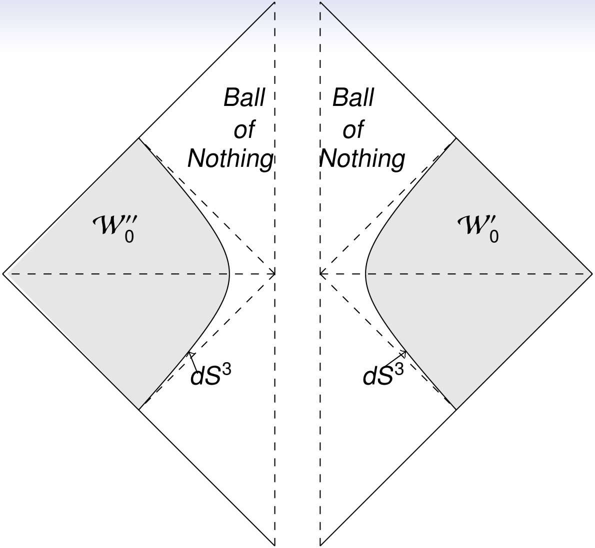


Figure: Penrose Conformal Diagram of the Hawking Wormhole. A ball of nothing is removed from two copies of the Minkowski spacetime. The two De Sitter boundaries dS^3 are identified to form the throat of the wormhole.

Geometry of the Wormhole

$$ds^2_{\mathcal{W}} = \left(1 + \frac{1}{4(\mid X \mid^2 - T^2)}\right)^2 \left\{dT^2 - dX^2\right\}$$

- Conformally flat
- Ricci Scalar $R = 0$.
- Violation of the Null Energy Condition

$$T_{\mu\nu} := \frac{1}{2} R g_{\mu\nu} - R_{\mu\nu}$$

$$V = \partial_0 \pm \partial_j$$

$$T_{\mu\nu} V^\mu V^\nu = \frac{-4(T \mp X^j)^2}{\left(\mid X \mid^2 - T^2 + \frac{1}{4}\right)^2 (\mid X \mid^2 - T^2)} < 0$$

$\rightarrow$ Exotic matter-energy...

Classical propagation : Heuristics

- Mechanical analogy :
  Bubble of Nothing $\leftrightarrow$ Expanding Accelerating Piston
  Time-like / Light-light geodesics $\leftrightarrow$ Massive / Massless particles
- Massless Particle : a unique collision.
  Massive Particle : infinity of collisions.
- Time-like geodesics : bounded and periodic.
  Light-like geodesics : go to infinity.
- Massive fields : asymptotically almost periodic profile.
  Massless fields : asymptotically free profile.

Issue : meaning of “massive” and “massless”.

Geodesics

- Let γ be a causal geodesic in $\mathcal{M}$. Then $\dot{t}(\lambda) \neq 0$, $t(\mathbf{R}) = \mathbf{R}$. $\omega(\mathbf{R})$ is included in a half of a great circle of S^2 and there exists $\lim_{\lambda \to \pm\infty} \omega(\lambda)$.
- Two points $(t_0, y_k, z_k; \omega_k) \in \mathbf{R}^3 \times S^2$, $k = 1, 2$, are causaly disconnected in the future if the distance between ω_1 and ω_2 on S^2 is larger than $2\pi - 4\arctan\left(e^{t_0}\right)$.
- If γ is time-like, then ρ is a bounded function of λ.
- If γ does not hit the bubble of nothing $\rho = R$, then $\psi(\mathbf{R}) = S^1$.
- If γ hits the bubble of nothing but does not stay on it, then $\psi(\mathbf{R})$ is a pair of two antipodal points. In this case $(y(\lambda), z(\lambda))$ delineates, either a whole straight line crossing $(0, 0)$ when γ is null, or a straight segment crossing the origin in its middle when γ is time-like, and then y and z are λ-periodic.
- There exists causal (null or time-like) geodesics that stay on the bubble of nothing.

Killing vectors fields ξ^μ :

$$\frac{\partial}{\partial\varphi},\ \frac{\partial}{\partial\psi},\ \cos\varphi\frac{\partial}{\partial t}-\tanh t\sin\varphi\frac{\partial}{\partial\varphi},\ \sin\varphi\frac{\partial}{\partial t}+\tanh t\cos\varphi\frac{\partial}{\partial\varphi}$$

Conserved currents $\xi_\mu\dot{x}^\mu$:

$$K_\varphi := \left(\rho^2\cosh^2 t\right)\dot{\varphi},\ \ K_\psi := \left(1-\frac{R^2}{\rho^2}\right)\dot{\psi}$$

$$K_t' = \rho^2\left(\dot{t}\cos\varphi+\dot{\varphi}\sinh t\cosh t\sin\varphi\right),\ K_t'' = \rho^2\left(\dot{t}\sin\varphi-\dot{\varphi}\sinh t\cosh t\cos\varphi\right)$$

$$\dot{\rho}^2+K_\psi^2+\left(1-\frac{R^2}{\rho^2}\right)\left[\frac{K_\varphi^2-K_t'^2-K_t''^2}{\rho^2}+E\right]=0,\ E=g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

Geodesics equation ($z=0$, $\theta=\pi/2$) :

$$\ddot{y}-\frac{\rho}{\rho+1}y\dot{y}^2W'\left({}^{+2}_{-2},y^2\right)+E\frac{y}{\rho}\left(\frac{\rho+1}{\rho+2}\right)^2=0,\ \ \rho=\frac{1}{2}W\left({}^{+2}_{-2},y^2\right)$$

Klein-Gordon equation

$$|\, det(g)\,|^{-\frac{1}{2}}\ \partial_\mu\left(|\, det(g)\,|^{\frac{1}{2}}\ g^{\mu\nu}\partial_\nu u\right) + M^2 u = 0, \quad M \geq 0$$

$$\left[\partial_t^2 + 2\tanh t\partial_t - \frac{1}{\cosh^2 t}\Delta_{S^2} + H_M\right] u = 0,$$

- Witten space-time (with $R = 1$, $\rho = \frac{1}{2}W\left({}^{+2}_{-2}, y^2 + z^2\right) = \cosh x$) :

$$H_M = -\frac{e^{2\rho}\rho}{(1+\rho)^2}\left[\partial_y\left(\rho^3\partial_y u\right) + \partial_z\left(\rho^3\partial_z u\right)\right] + M^2\rho^2 \ on\ \mathbf{R}^2_{y,z}$$

$$H_M = -\frac{1}{\sinh(2x)}\partial_x\left(\sinh(2x)\partial_x\right) - \frac{\cosh^4 x}{\sinh^2 x}\partial_\psi^2 + M^2\cosh^2 x \ \ on\ \ ]0,\infty[_x \times S^1_\psi,$$

- Hawking wormhole :

$$H_M = -\frac{1}{\cosh^2 x}\partial_x\left(\cosh^2 x\partial_x\right) + M^2\cosh^2 x \ \ on\ \ \mathbf{R}_x,$$

- dS^3 (Bubble of Nothing , Throat of Wormhole) :

$$H_M u = M^2 u$$

Energy, Witten space

$$\Sigma_t = \{t\} \times \mathbf{R}^2_{y,z} \times S^2 = \{t\} \times]0,\infty[_x \times S^1 \times S^2.$$

$$E(t) = \int_{\Sigma_t} \left[|\partial_t u|^2 + \frac{\rho^4 e^{2\rho}}{(1+\rho)^2} |\nabla_{\mathbf{R}^2} u|^2 + \frac{1}{\cosh^2 t} |\nabla_{S^2} u|^2 + M^2\rho^2 |u|^2 \right] d\mu$$

$$d\mu = \frac{(1+\rho)^2}{\rho} e^{-2\rho} dy\, dz\, d\Omega_2 = \frac{1}{2}\sinh(2x) dx\, d\psi\, d\Omega_2.$$

$$\frac{d}{dt}E(t) = -2\tanh t \int_{\Sigma_t} \left[2 |\partial_t u|^2 + \frac{1}{\cosh^2 t} |\nabla_{S^2} u|^2 \right] d\mu$$

Functional Framework, Witten space

$$\|u\|^2_{Y^1} := \int_\Sigma \left[\frac{\rho^4 e^{2\rho}}{(1+\rho)^2} \mid \nabla_{\mathbf{R}^2} u \mid^2 + \mid \nabla_{S^2} u \mid^2 + \rho^2 \mid u \mid^2 \right] d\mu$$

$$X^0 := L^2(\Sigma, d\mu), \quad Y^1 := \left\{ u \in X^0, \quad \|u\|_{Y^1} < \infty \right\}, \quad Y^1 \subset\subset X^0$$

$$\|u\|^2_{W^1} := \int_\Sigma \left[\frac{\rho^4 e^{2\rho}}{(1+\rho)^2} \mid \nabla_{\mathbf{R}^2} u \mid^2 + \mid \nabla_{S^2} u \mid^2 \right] d\mu, \quad W^1 \hookrightarrow X^0$$

$$X^0_0 := \left\{ u \in X^0, \ y\partial_z u - z\partial_y u = 0 \right\}, \quad X^0_\perp := \left(X^0_0 \right)^\perp, \quad W^1_* := W^1 \cap X^0_*, \ * = 0, \perp$$

$$W^1_\perp \hookrightarrow Y^1 \quad : \quad \int_\Sigma \rho^2 \mid u \mid^2 d\mu \leq \int_\Sigma \frac{\rho^4 e^{2\rho}}{(1+\rho)^2} \mid \nabla_{\mathbf{R}^2} u \mid^2 d\mu$$

$$W^1_0 \hookrightarrow X^0 \quad : \quad \int_\Sigma \left[1 + \frac{1}{4x^2} - \frac{1}{\sinh^2(2x)} \right] \mid u \mid^2 d\mu \leq \int_\Sigma \frac{\rho^4 e^{2\rho}}{(1+\rho)^2} \mid \nabla_{\mathbf{R}^2} u \mid^2 d\mu$$

Cauchy problem, Witten space

$$X^1 := W^1 \ \ when \ \ M = 0, \ \ X^1 := Y^1 \ \ when \ \ M > 0$$

$$X^1_0 := X^1 \cap X^0_0, \ \ X^1_\perp := X^1 \cap X^0_\perp$$

Existence and uniqueness of the solution of the global Cauchy Problem in

- $C^0\left(\mathbf{R}_t; X^1\right) \cap C^1\left(\mathbf{R}_t; X^0\right)$
- $C^0\left(\mathbf{R}_t; X^1_0\right) \cap C^1\left(\mathbf{R}_t; X^0_0\right)$
- $C^0\left(\mathbf{R}_t; X^1_\perp\right) \cap C^1\left(\mathbf{R}_t; X^0_\perp\right)$

Proof : Leray, Lions, Kato,....

Spectral analysis on $\mathbf{R}^2_{y,z}$

$$H_M = -\frac{e^{2\rho}\rho}{(1+\rho)^2}\left[\partial_y\left(\rho^3\partial_y u\right)+\partial_z\left(\rho^3\partial_z u\right)\right]+M^2\rho^2 \text{ on } \mathbf{R}^2_{y,z} \text{ with natural domain}$$

$$H_M = -\frac{1}{\sinh(2x)}\partial_x\left(\sinh(2x)\partial_x\right)-\frac{\cosh^4 x}{\sinh^2 x}\partial^2_\psi+M^2\cosh^2 x \quad \text{on} \quad]0,\infty[_x\times S^1_\psi$$

$$H_{M,n} = -\frac{1}{\sinh(2x)}\frac{d}{dx}\left(\sinh(2x)\frac{d}{dx}\right)+\frac{\cosh^4 x}{\sinh^2 x}n^2+M^2\cosh^2 x \quad \text{on} \quad]0,\infty[_x$$

Effective mass: $\mu := \sqrt{M^2+n^2}$. $\mu > 0 \Rightarrow \sigma(H_{M,n}) = \sigma_p(H_{M,n}) \subset]1,\infty[$

$$L = -\frac{1}{\sinh(2x)}\frac{d}{dx}\left(\sinh(2x)\frac{d}{dx}\right) \text{ on } L^2(]0,\infty[;\sinh(2x)dx)$$

$$\textit{Domain}: \quad u,\ u',\ Lu \in L^2(]0,\infty[;\sinh(2x)dx),\ \lim_{x\to 0} u'(x) = 0$$

L is selfadjoint, $\sigma(L) = \sigma_{ac}(L) = [1,\infty[$, generalized eigenfunctions

$$w(\lambda,;x) = \frac{2}{\pi\sinh(2x)}\tanh\left(\pi\sqrt{\lambda-1}\right)\mathbf{Q}^{i\sqrt{\lambda-1}}_{-\frac{1}{2}}(\coth(2x))$$

Klein-Gordon equation in dS^3

$$\partial_t^2 v + 2\tanh t \partial_t v - \frac{1}{\cosh^2 t}\Delta_{S^2} v + \lambda v = 0 \ \ on \ \ \mathbf{R}_t \times S^2, \ \ \lambda > 1$$

$$v \in C^0(\mathbf{R}_t, H^1(S^2)) \cap C^1(\mathbf{R}_t, L^2(S^2)), \ v(t,\Omega_2) = \sum_{l,m} v_{l,m}(t) Y_{l,m}(\Omega_2)$$

$$v'' + 2\tanh(t) v' + \frac{l(l+1)}{\cosh^2 t} v + \lambda v = 0$$

$$w(t) := (\cosh t) v(t)$$

$$w'' + (\lambda - 1) w + \frac{l(l+1)}{\cosh^2 t} w = 0 \ \ (\text{Pöschl} - \text{Teller})$$

$$w(t) = A^+ \mathrm{P}_l^{i\sqrt{\lambda-1}}(\tanh t) + A^- \mathrm{P}_l^{i\sqrt{\lambda-1}}(-\tanh t)$$

$$A^{\pm} = \frac{2^{-i\sqrt{\lambda-1}}}{2\sqrt{\pi}} \Bigg[\Gamma\left(\frac{l}{2} + 1 - \frac{i}{2}\sqrt{\lambda-1}\right)\Gamma\left(\frac{1}{2} - \frac{l}{2} - \frac{i}{2}\sqrt{\lambda-1}\right) w(0)$$
$$\pm \frac{1}{2}\Gamma\left(\frac{l+1}{2} - \frac{i}{2}\sqrt{\lambda-1}\right)\Gamma\left(-\frac{l}{2} - \frac{i}{2}\sqrt{\lambda-1}\right) w'(0) \Bigg].$$

Kaluza-Klein Tower in Witten space

$$\mu > 0 \Leftrightarrow M > 0 \;\; or \;\; u \in C^0(\mathbf{R}_t; X^0_\perp)$$

$$u(t,x,\Omega_2,\psi) = \sum_{k,l,m,\pm} \frac{A^{\pm}_{k,l,m,n}}{\cosh t} \mathrm{P}_l^{i\sqrt{\lambda_{n,k}-1}}(\pm \tanh t) Y_{l,m}(\Omega_2) w_{n,k}(x) e^{in\psi},$$

$$\left[-\frac{1}{\sinh(2x)}\frac{d}{dx}\left(\sinh(2x)\frac{d}{dx}\right) + \frac{\cosh^4 x}{\sinh^2 x} n^2 + M^2 \cosh^2 x\right] w_{n,k} = \lambda_{n,k} w_{n,k}$$

$$\mu = 0 \Leftrightarrow M = 0 \;\; and \;\; u \in C^0(\mathbf{R}_t; X^0_0)$$

$$u = \frac{2}{\pi} \lim_{A\to\infty} \left(\frac{1}{\sinh(2x)}\right)^{\frac{1}{2}} \int_1^A \hat{u}(\lambda; t, \Omega_2) \tanh\left(\frac{\pi}{2}\sqrt{\lambda-1}\right) \boldsymbol{Q}_{-\frac{1}{2}}^{\frac{i}{2}\sqrt{\lambda-1}}(\coth(2x))\, d\lambda$$

$$\hat{u}(\lambda; t, \Omega_2) = \frac{1}{\cosh t} \sum_{l,m,\pm} A^{\pm}_{l,m}(\lambda) \mathrm{P}_l^{i\sqrt{\lambda-1}}(\pm \tanh t) Y_{l,m}(\Omega_2),$$

Scattering

Profile : $v(t,.) := \cosh(t)u(t,.)$, $Pv = 0$:

$$P = \partial_t^2 - \frac{1}{\sinh(2x)}\partial_x\left(\sinh(2x)\partial_x\right) - \frac{1}{\cosh^2 t}\Delta_{S^2} - \frac{\cosh^4 x}{\sinh^2 x}\partial_\psi^2 + M^2\cosh^2 x - 1$$

$$P_\infty = \partial_t^2 - \frac{1}{\sinh(2x)}\partial_x\left(\sinh(2x)\partial_x\right) - \frac{\cosh^4 x}{\sinh^2 x}\partial_\psi^2 + M^2\cosh^2 x - 1$$

$$E_\infty(v,t) = \int_{\Sigma_t}\left[\mid \partial_t v \mid^2 + \mid \partial_x v \mid^2 + \frac{\cosh^4 x}{\sinh^2 x}\mid \partial_\psi v \mid^2 + (M^2\cosh^2 x - 1)\mid v \mid^2\right] d\mu$$

- Existence of symptotic profiles $v_{in(out)}$,

$$P_\infty v_{in(out)} = 0, \quad v(t) \sim v_{in(out)}(t), \quad t \to \pm\infty$$

- Wave Operators $\Omega_{in(out)} : \; v \mapsto v_{in(out)}$
- Scattering Operator $S : \; v_{in} \mapsto v_{out}$
- Quantization

Asymptotics in dS^3

$$w(t) := A^+ \mathrm{P}_l^{i\sqrt{\lambda}}(\tanh t) + A^- \mathrm{P}_l^{i\sqrt{\lambda}}(-\tanh t), \quad \lambda > 0, \quad l \in \mathbf{N}$$

$$\left| w(t) - \left(w^+_{in(out)} e^{it\sqrt{\lambda}} + w^-_{in(out)} e^{-it\sqrt{\lambda}} \right) \right| \to 0, \; t \to -\infty \; (t \to +\infty)$$

$$w^{-(+)}_{in(out)} = \frac{1}{\Gamma(1 - i\sqrt{\lambda})} A^{-(+)}, \quad w^{+(-)}_{in(out)} = (-1)^l \frac{\Gamma(l + 1 + i\sqrt{\lambda}))}{\Gamma(l + 1 - i\sqrt{\lambda})\Gamma(1 + i\sqrt{\lambda})} A^{+(-)}$$

NO REFLECTION : $\quad w^{\pm}_{out} = (-1)^l \dfrac{\Gamma(1 \pm i\sqrt{\lambda})}{\Gamma(1 \mp i\sqrt{\lambda})} \dfrac{\Gamma(l + 1 \mp i\sqrt{\lambda})}{\Gamma(l + 1 \pm i\sqrt{\lambda})} w^{\pm}_{in}$

$$\left(\tanh \sqrt{\lambda}\right) \left[(l + 1 + \sqrt{\lambda}) \mid w(0) \mid^2 + (l + 1 + \sqrt{\lambda})^{-1} \mid w'(0) \mid^2 \right] \lesssim$$

$$\lesssim \sqrt{\lambda} \left[\mid w^+_{in(out)} \mid^2 + \mid w^-_{in(out)} \mid^2 \right] \lesssim$$

$$\lesssim \left(\coth \sqrt{\lambda}\right) \left[(l + 1 + \sqrt{\lambda}) \mid w(0) \mid^2 + (l + 1 + \sqrt{\lambda})^{-1} \mid w'(0) \mid^2 \right]$$

Theorem

Let v be a solution in $C^0(\mathbf{R}_t; X^1) \cap C^1(\mathbf{R}_t; X^0)$.

- *There exists unique $v_{in(out)} \in C^0(\mathbf{R}_t; \dot{X}^1) \cap C^1(\mathbf{R}_t; X^0)$ such that*

$$E_\infty\left(v - v_{in(out)}, t\right) \to 0, \quad t \to -(+)\infty.$$

- *The wave operators $v \mapsto v_{in(out)}$ are one-to-one and we have*

$$E_\infty\left(v_{out}\right) = E_\infty\left(v_{in}\right).$$

- *Effective mass > 0 : v is asymptotically almost periodic*
- *Effective mass $= 0$: v is dispersive :*

$$\sqrt{\frac{\sinh(2 \mid \mathbf{x} \mid)}{\mid \mathbf{x} \mid}}\, v(t, x = \mid \mathbf{x} \mid, \Omega_2) \sim W_{in(out)}(t, \mathbf{x}, \Omega_2), \quad t \to -(+)\infty,$$

$$\partial_t^2 W_{in(out)} - \Delta_{\mathbf{R}^2} W_{in(out)} = 0 \text{ in } \mathbf{R}_t \times \mathbf{R}^2_{\mathbf{x}} \times S^2$$

Quantization

$$v \leftrightarrow v_{l,m}(\lambda),\ (l,m) \in \mathcal{I},\ \lambda \in \Lambda,\ \mu > 0 \Rightarrow \Lambda\ discret,\ \mu = 0 \Rightarrow \Lambda = [a, \infty[$$

$$(v, \partial_t v) \in \mathcal{E} := L^2(\mathcal{I} \times \Lambda; \mid \lambda \mid^{\frac{1}{2}} \delta_{l,m} \otimes d\lambda) \times L^2(\mathcal{I} \times \Lambda; \mid \lambda \mid^{-\frac{1}{2}} \delta_{l,m} \otimes d\lambda)$$

$$\sigma\left((v_1, v_1'); (v_2, v_2')\right) = \int_{\mathcal{I}\times\Lambda} \overline{v_1} v_2' - \overline{v_1'} v_2\ \delta_{l,m} d\lambda$$

$$\mathcal{E} = \mathcal{E}_{pos} \oplus \mathcal{E}_{neg},\quad (v, v') \in \mathcal{E}_{pos(neg)},\quad U(t)(v, v') = e^{+(-)i|\lambda|^{\frac{1}{2}}t}(v, v')$$

$$(v_{in}, v_{in}') \in \mathcal{E}_{pos(neg)},\quad (v_{out}, v_{out}') = S((v_{in}, v_{in}')) \in \mathcal{E}_{pos(neg)}.$$

$$v_{out;l,m}(\lambda) = (-1)^l \frac{\Gamma(1 + (-)i\sqrt{\lambda})}{\Gamma(1 - (+)i\sqrt{\lambda})} \frac{\Gamma(l + 1 - (+)i\sqrt{\lambda})}{\Gamma(l + 1 + (-)i\sqrt{\lambda})} v_{in;l,m}(\lambda)$$

Theorem

The Scattering operator S is unitarily implementable in the Fock-Cook quantization of $(\mathcal{E}, U, \sigma)$, and the quantized Scattering operator lets invariant the Fock vacuum.

NO CREATION OF PARTICLE !

Resonances

Effective Mass $\mu = 0$

Scattering Amplitude $S_{l,m}(\lambda)$, $\lambda \in (0, \infty)$

$$S_{l,m} : \quad \sqrt{\lambda} \longmapsto (-1)^l \frac{\Gamma(1 + (-) i\sqrt{\lambda})}{\Gamma(1 - (+) i\sqrt{\lambda})} \frac{\Gamma(l + 1 - (+) i\sqrt{\lambda})}{\Gamma(l + 1 + (-) i\sqrt{\lambda})}$$

Simple Pole $\sqrt{\lambda} = +(-)(n+1)i, \quad 0 \leq n < l.$

Vanishing profiles :

$$v(t, x, \Omega_2) = \mathrm{P}_l^{\pm i(n+1)}(\tanh t) Y_{l,m}(\Omega_2) \boldsymbol{Q}_{-\frac{1}{2}}^{\pm\frac{n+1}{2}}(\coth(2x))$$

$$\left| \mathrm{P}_l^{i\sqrt{\lambda}}(\pm \tanh t) \right| \lesssim (\cosh t)^{-n-1}.$$

Hawking Wormhole

profile $v(t,x,\Omega_2) := \cosh(t)\cosh(x)u(t,x,\Omega_2)$,

$$\left[\partial_t^2 - \partial_x^2 - \frac{1}{\cosh^2 t}\Delta_{S^2} + M^2\cosh^2 x\right] v = 0, \quad x \in \mathbf{R}$$

$$0 < M, \quad w_k'' + M^2(\cosh^2 x)w_k = \lambda_k w_k$$

$$v(t,x,\Omega_2) = \sum_{k,l,m,\pm} A^{\pm}_{k,l,m} w_k(x) Y_{l,m}(\Omega_2) \mathrm{P}_l^{i\sqrt{\lambda_k}}(\pm\tanh t)$$

$$M = 0, \quad v(t,x,\Omega_2) = \frac{1}{2\pi}\sum_{l,m,\pm} Y_{l,m}(\Omega_2) \lim_{A\to\infty}\int_{-A}^{A} e^{ix\xi}\mathrm{P}_l^{i|\xi|}(\pm\tanh t)A^{\pm}_{l,m}(\xi)d\xi,$$

$$T = e^{|x|}\sinh t, \quad X = e^{|x|}\cosh t\Omega_2, \quad u_\pm(T,X) = \left(1 + e^{-2|x|}\right)u(t,x,\Omega_2), \quad \pm x \geq 0.$$

$$\partial_T^2 u_\pm - \Delta_{\mathbf{R}^3_X} u_\pm = 0 \;\; in \;\; |X|^2 - T^2 > 1/4$$

$$u_+ = u_-, \quad T\partial_T u_+ + X.\nabla_X u_+ = -T\partial_T u_- - X.\nabla_X u_- \;\; when \;\; |X|^2 - T^2 = 1/4.$$

Scattering by Wormhole

$$\left[\partial_t^2 - \partial_x^2 + M^2 \cosh^2 x\right] v_{in(out)} = 0, \ \ in\ \mathbf{R}_t \times \mathbf{R}_x \times S^2$$

$$E_\infty(v) := \int_{\mathbf{R}\times S^2} \mid \nabla_{t,x} v \mid^2 + M^2 \cosh^2(x) \mid v \mid^2 \, dx d\Omega_2.$$

Theorem

$$E_\infty(v(t) - v_{in(out)}(t)) \to 0, \ \ t \to -(+)\infty, \ \ E_\infty(v_{in}) = E_\infty(v_{out})$$

- $0 < M$: v is asymptotically almost periodic
  Massive fields trapped near the throat : wormhole NOT traversable by the massive fields.
- $M = 0$: $v(t, x, \Omega_2) \sim v^+_{in(out)}(x + t, \Omega_2) + v^-_{in(out)}(x - t, \Omega_2)$
  The wormhole IS TRAVERSABLE by the massless fields.
- Diagonal Unitary Scattering Operator : NO mixing positive/negative frequencies.
- Quantum transparence : NO creation of particle.

Concluding Remarks

- Stability of the Witten solution.
- Kerr Bubble of Nothing (disappearing).
- Bubble of Nothing and Wormhole in Brane Cosmology.
- Bubble(s) of Nothing + Black-Hole(s) / Black-String(s) solution.
- Bubble Bath Solution.
- Detector of particles.