Simplification of the Keiper/Li approach to the Riemann Hypothesis (RH)

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NOT ABOUT - “proofs” of RH
- importance of RH for number theory

ONLY ABOUT
- criteria (rewordings, equivalent statements, tests) for RH
The Riemann zeta function $\zeta(x)$

$$
\zeta(x) \overset{\text{def}}{=} \sum_{k=1}^{\infty} k^{-x} \quad \text{ (Re } x > 1) \\
\equiv \prod_{\{p\}} (1 - p^{-x})^{-1} \quad \text{ (Euler product: i.e., log } \zeta \text{ encodes the primes)} \\
\equiv \frac{1}{\Gamma(x)} \int_{0}^{\infty} \frac{1}{e^u - 1} u^{x-1} \, du \quad \text{ (Mellin transformation)}
$$

which extends $\zeta$ to $\mathbb{C}$ with the only singularity $\zeta(x) = \frac{1}{x - 1} + \ldots$, and

$$
\zeta(-n) = (-1)^n B_{n+1}/(n + 1) \quad \text{ for } n = 0, 1, 2, \ldots
$$

(Poisson’s summation formula $\Rightarrow$) **Riemann’s Functional Equation:**

$$
\xi(x) \equiv \xi(1 - x) \quad \text{ for } 2\xi(x) \overset{\text{def}}{=} x(x-1)\pi^{-x/2}\Gamma(x/2) \zeta(x) \text{ (completed zeta function).}
$$

$\xi$ is an entire function, with these **symmetries** besides $x \leftrightarrow (1 - x) : \text{Im} \leftrightarrow -\text{Im}$ (reality), hence also symmetry / **critical line** $\{\text{Re } x = \frac{1}{2}\}$. 
Riemann zeros = zeros of $\xi(x)$ (denoted $\{\rho\}$), the lowest ones being

$$\rho = \frac{1}{2} \pm iT, \quad T \approx 14.1347251, 21.0220396, 25.0108576, 30.4248761, 32.9350616, \ldots$$
The Riemann zeros - proven facts

• Countably many zeros, all in the open *critical strip* \( \{0 < \Re x < 1\} \).

\[
\begin{align*}
\mathfrak{N}(T) &= \frac{T}{2\pi} \left( \log \frac{T}{2\pi} - 1 \right) + \delta N(T), \quad \delta N(T) = O(\log T) \text{ as } T \to +\infty.
\end{align*}
\]

• *Riemann–von Mangoldt theorem* for the zeros’ counting function \( \mathcal{N}(T) \) :

\[
2\xi(x) = \prod_{\rho \neq 0, 1 - \rho} (1 - x/\rho).
\]
The Riemann Hypothesis (RH) (1859)

All the zeros $\rho$ of $\xi(x)$ lie on the axis $\{\text{Re } x = \frac{1}{2}\}$ (the critical line)

Numerically verified up to the $10^{13}$-th zero or the height $T_0 \approx 2.4 \cdot 10^{12}$ (Gourdon 2004).

Significance for number theory: the Prime Number Theorem for the primes’ counting function $\pi(k)$ states

$$\pi(k) \sim \text{Li}(k) \overset{\text{def}}{=} \int_2^k \frac{dv}{\log v} \left( \approx \frac{k}{\log k} \right) \quad \text{for } k \to \infty.$$ 

Then, RH optimizes the error term $\varepsilon(k) = \pi(k) - \text{Li}(k)$:

$$\beta_0 \overset{\text{def}}{=} \sup_{\rho} \{\text{Re } \rho\} \equiv \inf\{\beta \in \mathbb{R} \mid \varepsilon(k) = O(k^{\beta})\},$$

and RH ($\iff \beta_0 = \frac{1}{2}$) more precisely amounts to $\varepsilon(k) = O(k^{1/2} \log k)$: the least possible fluctuation for the primes’ distribution function $\pi(k)$ around its mean $\text{Li}(k)$.

But currently: RH true or not? $\beta_0 < 1$ or $\beta_0 = 1$?
The Keiper/Li sequence as a testing tool for RH

Defined by the (real) generating function (Keiper 1992, X.-J. Li 1997)

\[ \lambda_n z^{n-1} \equiv F(x(z)) \overset{\text{def}}{=} \frac{d}{dz} \log 2\xi\left(x = \frac{1}{1 - z}\right) \iff \lambda_n = \sum_{\langle \rho, 1 - \rho \rangle} \left[ 1 - (1 - 1/\rho)^n \right]. \]

\[ \lambda_1 = 1 - \frac{1}{2} \log 4\pi + \frac{1}{2} \gamma \approx 0.0230957, \quad \lambda_2 \approx 0.0923457, \quad \lambda_3 \approx 0.207639, \ldots \]

RH \iff all \( z_\rho = 1 - 1/\rho \) on \( \{|z| = 1\} \).
\[ x = \frac{1}{1 - z} \quad \iff \quad z = 1 - \frac{1}{x} \]
\[ x = \frac{1}{2} + iT \quad \iff \quad z = e^{i\theta}, \quad T \equiv \frac{1}{2} \cot \frac{1}{2} \theta. \]
\[ \rho = \frac{1}{2} \pm iT_\rho \quad \iff \quad z_\rho = e^{\pm i\theta_\rho}; \quad \text{Re } \rho = \frac{1}{2} \iff \theta_\rho \text{ real.} \]

\[ \lambda_n = \sum_{\langle \rho, 1 - \rho \rangle} \left[ 1 - (1 - 1/\rho)^n \right] \equiv \sum_{\langle \rho, 1 - \rho \rangle} (1 - z_\rho^n) \equiv \sum_{\langle \rho, 1 - \rho \rangle} (1 - \cos n\theta_\rho). \]
If RH is true...

\[ \lambda_n = \sum_{\langle \rho, 1-\rho \rangle} (1 - \cos n\theta_\rho) \quad \text{with all } \theta_\rho \text{ real} \]

implies, for the \( \lambda_n \),

- **positivity:** \( \lambda_n > 0 \) for all \( n \) (Keiper 1992)
- **large-order behavior through the integral formula** (Oesterlé 2000)

\[ \lambda_n = 2 \int_0^\infty (1 - \cos n\theta) \, dN(T) \quad \implies \quad \frac{\lambda_n}{n} = 2 \int_0^\pi \sin n\theta N\left(\frac{1}{2} \cot \frac{1}{2}\theta\right) \, d\theta \]

and using \( N(T) = \frac{T}{2\pi} \left( \log \frac{T}{2\pi} - 1 \right) + \delta N(T) \), \( \delta N(T) = O(\log T)_{T \to +\infty} \);

- \( \int_0^\pi \sin n\theta \, dN\left(\frac{1}{2} \cot \frac{1}{2}\theta\right) \, d\theta \to 0 \) (Riemann–Lebesgue lemma)
- \( \frac{\lambda_n}{n} \sim 2 \int_0^\infty \sin \Theta \frac{n}{2\pi\Theta} \left[ \log \frac{n}{2\pi\Theta} - 1 \right] \frac{d\Theta}{n} \quad \text{using } \Theta \overset{\text{def}}{=} n\theta, \ T \sim n/\Theta. \)
- \( \int_0^\infty \sin \Theta \, d\Theta = \frac{1}{2}\pi \), \( \int_0^\infty \sin \Theta \log \Theta \, d\Theta = -\frac{1}{2}\pi \gamma \quad \implies \)

\[ \frac{\lambda_n}{n} \sim \frac{1}{2} \log n + \frac{1}{2} (\gamma - \log 2\pi - 1) \]
Asymptotic sensitivity to RH

Large-order behavior of \( \lambda_n \) : using Darboux's method for \( F(x(z)) \), a meromorphic function having simple poles at all images \( z_\rho \) of Riemann zeros,

\[
\lambda_n = \frac{1}{2\pi i} \oint_C z^{-n} F(x(z)) \, dz = - \sum_{\{ |z_\rho| < 1 \}} {z_\rho}^{-n} + o(r^{-n})_{n \to \infty} \text{ for all } r < 1.
\]
Two concrete criteria for RH

- Li’s criterion (X.-J. Li 1997): \( \text{RH true} \iff \lambda_n > 0 \) for all \( n \)

- Asymptotic alternative (AV 2004, by saddle-point method on \( \lambda_n \) as an integral)

\[
\lambda_n \sim \begin{cases} 
\sum_{\{|z_\rho|<1\}} z_\rho^{-n} + o(r^{-n})_{n\to\infty} & \text{for all } r < 1 \\
\text{(exponentially growing oscillations of both signs)} \\
\frac{1}{2}n \log n + \frac{1}{2}n(\gamma - \log 2\pi - 1) + O(n^{1/2} \log n) & \text{if RH true}
\end{cases}
\]

(lagarias 2007)

Practical aspects:

- RH verified up to a height \( T_0 \implies \lambda_n > 0 \) as long as \( n < T_0^2 \)

- A Riemann zero violating RH, \( \rho = \frac{1}{2} + t \pm iT \) with \( 0 < t < \frac{1}{2} \) (hence \( T > T_0 \)), will practically be undetectable through \( \lambda_n \) unless

\[
n \gtrsim T^2/t > 2T^2 \quad (\gtrsim 10^{25} \text{ currently: } T_0 \approx 2.4 \cdot 10^{12}).
\]
Computing the $\lambda_n$

$$\lambda_n = 1 - \frac{1}{2} (\log 4\pi + \gamma)n + \sum_{j=2}^{n} (-1)^j \binom{n}{j} (1 - 2^{-j}) \zeta(j) - \sum_{j=1}^{n} \binom{n}{j} \eta_{j-1}$$

(Bombieri–Lagarias 1999)

$$\log[(x - 1)\zeta(x)] \equiv -\sum_{n=0}^{\infty} \frac{\eta_n}{n+1}(x - 1)^{n+1}$$

vs

$$\log[(x - 1)\zeta(x)] \equiv \sum_{n=0}^{\infty} \gamma_n(x - 1)^{n+1}$$

(Sieltjes constants: $\gamma_0 \equiv \gamma$)

In terms of the von Mangoldt function $\Lambda(k) \overset{\text{def}}{=} \begin{cases} \log p & \text{if } k = p^r, \ p \text{ prime} \\ 0 & \text{otherwise} \end{cases}$

$$\eta_n = \frac{(-1)^n}{n!} \lim_{K \to \infty} \left\{ \sum_{k=2}^{K} \frac{\Lambda(k)(\log k)^n}{k} - \frac{(\log K)^{n+1}}{n+1} \right\} \quad (\eta_0 \equiv -\gamma).$$
Numerical computations (also: Coffey 2005)

\[ \lambda_n = n \log n + \frac{1}{2} n (\gamma - \log 2\pi - 1) \]  

Maślanka 2004

\[ \eta_0 = -\gamma_0 \]
\[ \eta_1 = +\gamma_0^2 - 2\gamma_1 \]
\[ \eta_2 = -\gamma_0^3 + 3\gamma_0\gamma_1 - 3\gamma_2 \]
\[ \eta_3 = +\gamma_0^4 - 4\gamma_0^2\gamma_1 + 2\gamma_1^2 + 4\gamma_0\gamma_2 - 4\gamma_3 \]
\[ \eta_4 = -\gamma_0^5 + 5\gamma_0^3\gamma_1 - 5\gamma_0\gamma_1^2 + 5\gamma_0^2\gamma_2 + 5\gamma_0\gamma_3 - 5\gamma_4. \]

Keiper 1992
Deformations of the $\lambda_n$ : allowed?

$$\lambda_n = \frac{1}{(n-1)!} \left[ x^{n-1} \log 2\xi(x) \right]_{x=1}^{(n)} \iff \sum_{n=1}^{\infty} \lambda_n z^{n-1} = \frac{d}{dz} \log 2\xi \left( \frac{1}{1-z} \right)$$
Deformations of the $\lambda_n$ : allowed!

$$\lambda_n = \frac{1}{(n-1)!} \left[ x^{n-1} \log 2\xi(x) \right]_{x=1}^{(n)} \iff \sum_{n=1}^{\infty} \lambda_n z^{n-1} \equiv \frac{d}{dz} \log 2\xi \left( \frac{1}{1-z} \right)$$

$\lambda_n^{(b)} = \frac{2b+1}{(n-1)!} \left[ (x+b)^{n-1} \log 2\xi(x) \right]_{x=1+b}^{(n)} \iff \sum_{n=1}^{\infty} \lambda_n^{(b)} z^{n-1} \equiv \frac{d}{dz} \log 2\xi \left( \frac{bz+1+b}{1-z} \right)$

(Sekatskii 2014)
Numerical observations

\[ \frac{1}{2} (\log 2\xi)''(\frac{1}{2}) = 0.0231050 \ldots \]

\[ -4 \log 2\xi(\frac{1}{2}) = 0.0231003 \ldots \]

\[ (\log 2\xi)'(1) \equiv \lambda_1 = 0.0230957089661210 \ldots \]
We propose broader deformations

\[
\lambda_n = \frac{1}{(n-1)!} [x^{n-1} \log 2\xi(x)]^{(n)} \bigg|_{x=1} \iff \sum_{n=1}^{\infty} \lambda_n z^{n-1} \equiv \left. \frac{d}{dz} \log 2\xi(x(z)) \right|_{z=1}.
\]

Denominator:

\[
z^n
\]
We propose broader deformations

\[ \lambda_n = \frac{1}{(n-1)!} \left[ x^{n-1} \log 2\xi(x) \right]^{(n)} \bigg|_{x=1} \]

\[ \lambda_n = \frac{1}{2\pi i} \oint_C \frac{dz}{z^n} F(x(z)), \quad \iff \quad \sum_{n=1}^{\infty} \lambda_n z^{n-1} \equiv \frac{F}{1 - z} \]

Denominator:

\[ z^n \implies \text{New denominator } (z - z_1) \cdots (z - z_n) \]
Asymptotic alternative (AV 2015)

\[ \Lambda_n \sim \begin{cases} 
\sum_{\{\text{Re } \rho > 1/2\}} e^{i\phi(\rho)} \frac{(-1)^n(2n)^{\text{Re } \rho - 1/2}}{|\text{Im } \rho|^{\text{Re } \rho + 3/2} \log n} & \text{if RH false} \\
\text{(power-like oscillations of both signs)} & \\
\log n + \frac{1}{2}n(\gamma - \log \pi - 1) + o(1) & \text{if RH true} \\
\text{(tempered growth to } +\infty) & 
\end{cases} \]

\[ \lambda_n = \frac{1}{2\pi i} \oint_C \frac{F}{z^n} \, dz \sim - \sum_{\{|z_\rho| < 1\}} z_\rho^{-n}; \quad \Lambda_n \sim \frac{1}{2\pi i} \int_{C_n} G n^{x - 1/2} \, dx \sim \sum_{\{\text{Re } \rho > 1/2\}} e^{i\phi(\rho)} \frac{(-1)^n(2n)^t}{|T|^{t+2} \log n} \]

0-17
Numerical computation of the $\Lambda_n$

Mathematica 10.3.0 for Linux x86 (64-bit)
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In[1]:= p := Floor[0.76555 n] + 16

In[2]:= n = 1000

Out[2]= 1000

In[3]:= Sum[(-1)^m N[ (2 (m+n)-1)!! / (m! (n-m)! (2m-1)!! (2m-1))
(\[Log]\[Abs][BernoulliB[2m]] - \[Log][2m-3]!!) ] / (-2)^n p], {m, n}] +
N[ (1-(-2)^n n!/(2n-1)!!) \[Log][2 Pi]/2, p]

\( \Lambda_n \) computed up to \( n = 4000 \), on a logarithmic \( n \)-scale; straight line: the asymptotic form

\[
\bar{\Lambda}_n = \log n + \frac{1}{2} (\gamma - \log \pi - 1) \approx \log n - 0.78375711.
\]

\( \Lambda_1 = \frac{3}{2} \log \frac{\pi}{3} \approx 0.0691764, \quad \Lambda_2 \approx 0.2274543, \quad \Lambda_3 \approx 0.4567141 \)
Adjustable precision $p(m) \approx \log_{10} |s_{nm}| + D$ 

In[1]:= p := \text{Floor}[0.76555 \ n] + 16

$$\log_{10} |s_{nm}| \sim \left[ -2 \frac{m}{n} \log_{10} \frac{m}{n} + \left(1+\frac{m}{n}\right) \log_{10} \left(1+\frac{m}{n}\right) - \left(1-\frac{m}{n}\right) \log_{10} \left(1-\frac{m}{n}\right) \right] n$$
The remainder sequence \( \delta \Lambda_n = \Lambda_n - \bar{\Lambda}_n \) (in gray), and a rectified form \((-1)^n \delta \Lambda_n\) (black dots) to cancel the period-2 oscillations.

Averaged and rescaled remainder sequence \( n \, \overline{\delta \Lambda}_n \) \( \text{def} = \frac{1}{2} n (\delta \Lambda_n + \delta \Lambda_{n-1}) \).