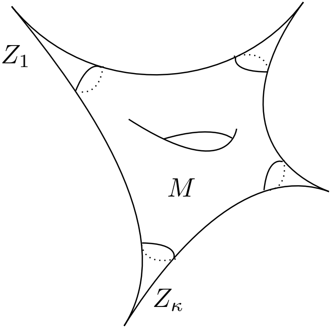
Weyl Laws for manifolds with cusps.

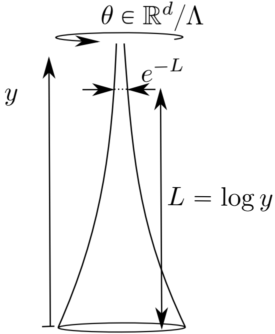
Some theorems funded by Université d'Orsay and CRM (Montréal).

What is a manifold with cusps ?



$\mathrm{vol}(M) < \infty$

(M, g) is complete.



Exact hyperbolic cusp

$g = \frac{dy^2 + d\theta^2}{y^2}$

$K = -1$

The free 1 body stationary problem: find solutions to the equation

$$-\Delta u = Eu. \qquad (*)$$

The free 1 body stationary problem: find solutions to the equation

$$-\Delta u = Eu. \tag{$*$}$$

Problem: How many solutions are there to $(*)$ with $E \leqslant \lambda^2$, on a cusp manifold ?

The free 1 body stationary problem: find solutions to the equation

$$-\Delta u = Eu. \tag{$*$}$$

Problem: How many solutions are there to $(*)$ with $E \leqslant \lambda^2$, on a cusp manifold ?
Why cusp manifolds ?

The free 1 body stationary problem: find solutions to the equation

$$-\Delta u = Eu. \tag{$*$}$$

Problem: How many solutions are there to $(*)$ with $E \leqslant \lambda^2$, on a cusp manifold ?
Why cusp manifolds ?

- If (M, g) is a compact riemannian $(d+1)$-manifold, we have the Weyl law:

  $$\#\{\text{ solutions with } E \leqslant \lambda^2\} \sim \frac{\mathrm{vol}(B^*M)}{(2\pi)^{d+1}}\lambda^{d+1}.$$

  The right hand side still makes sense

The free 1 body stationary problem: find solutions to the equation

$$-\Delta u = Eu. \tag{$*$}$$

Problem: How many solutions are there to $(*)$ with $E \leqslant \lambda^2$, on a cusp manifold ?
Why cusp manifolds ?

- If (M, g) is a compact riemannian $(d+1)$-manifold, we have the Weyl law:

  $$\#\{\text{ solutions with } E \leqslant \lambda^2\} \sim \frac{\mathrm{vol}(B^*M)}{(2\pi)^{d+1}} \lambda^{d+1}.$$

  The right hand side still makes sense
- On a non-compact manifold, in general, counting estimates for solutions of $(*)$ involve the dimension of the trapped set. Cusp manifold are special since the *trapped* set has full dimension.

The free 1 body stationary problem: find solutions to the equation

$$-\Delta u = Eu. \qquad (*)$$

Problem: How many solutions are there to $(*)$ with $E \leqslant \lambda^2$, on a cusp manifold ?
Why cusp manifolds ?

- If (M, g) is a compact riemannian $(d+1)$-manifold, we have the Weyl law:

  $$\#\{\text{ solutions with } E \leqslant \lambda^2\} \sim \frac{\mathrm{vol}(B^* M)}{(2\pi)^{d+1}} \lambda^{d+1}.$$

  The right hand side still makes sense
- On a non-compact manifold, in general, counting estimates for solutions of $(*)$ involve the dimension of the trapped set. Cusp manifold are special since the *trapped* set has full dimension.
- Hyperbolic cusps are simple models of finite volume ends.

The free 1 body stationary problem: find solutions to the equation

$$-\Delta u = Eu. \tag{$*$}$$

Problem: How many solutions are there to $(*)$ with $E \leqslant \lambda^2$, on a cusp manifold ?
Why cusp manifolds ?

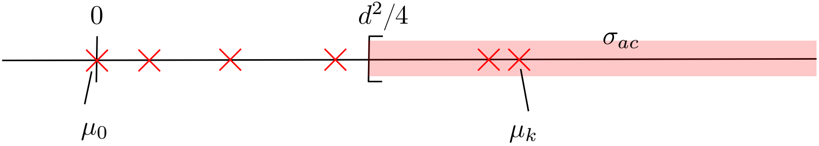
- If (M, g) is a compact riemannian $(d+1)$-manifold, we have the Weyl law:

$$\#\{\text{ solutions with } E \leqslant \lambda^2\} \sim \frac{\mathrm{vol}(B^*M)}{(2\pi)^{d+1}}\lambda^{d+1}.$$

  The right hand side still makes sense
- On a non-compact manifold, in general, counting estimates for solutions of $(*)$ involve the dimension of the trapped set. Cusp manifold are special since the *trapped* set has full dimension.
- Hyperbolic cusps are simple models of finite volume ends.
- They appear in other contexts (Arithmetics — recall the Riemann zeta function, geometry of the Teichmüller space).

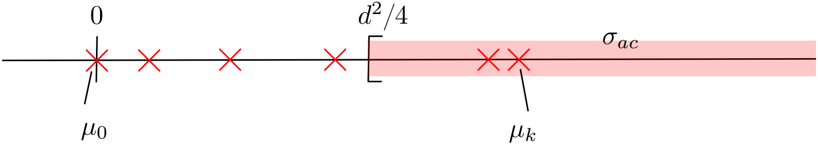
What should one count ?

The spectrum of $-\Delta$ on $L^2(M)$ splits into $\sigma_{pp} = \{\mu_0 = 0 < \mu_1 < \ldots\}$ and $\sigma_{ac} = [d^2/4, +\infty[$.



What should one count ?

The spectrum of $-\Delta$ on $L^2(M)$ splits into $\sigma_{pp} = \{\mu_0 = 0 < \mu_1 < \dots\}$ and $\sigma_{ac} = [d^2/4, +\infty[$.

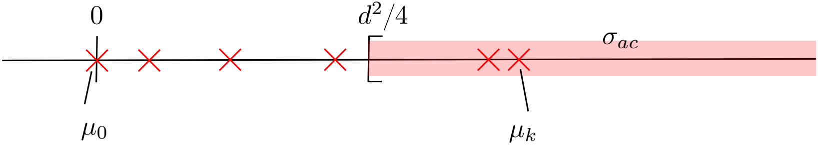


Let

$$N_{pp}(\lambda) = \#\{\mu_i \mid \mu_i \leqslant d^2/4 + \lambda^2\}.$$

What should one count ?

The spectrum of $-\Delta$ on $L^2(M)$ splits into $\sigma_{pp} = \{\mu_0 = 0 < \mu_1 < \dots\}$ and $\sigma_{ac} = [d^2/4, +\infty[$.



Let

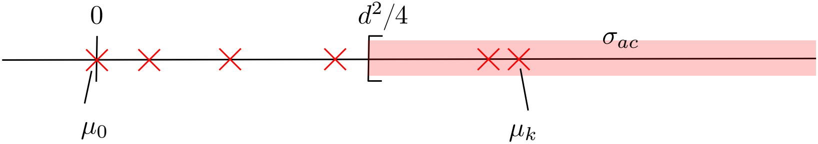
$$N_{pp}(\lambda) = \#\{\mu_i \mid \mu_i \leqslant d^2/4 + \lambda^2\}.$$

Theorem[Colin de Verdière, 83] N_{pp} is a constant for generic smooth metrics.

$\Longrightarrow$ Counting L^2 solutions to $(*)$ may not be very interesting.

What should one count ?

The spectrum of $-\Delta$ on $L^2(M)$ splits into $\sigma_{pp} = \{\mu_0 = 0 < \mu_1 < \dots\}$ and $\sigma_{ac} = [d^2/4, +\infty[$.



Let

$$N_{pp}(\lambda) = \#\{\mu_i \mid \mu_i \leqslant d^2/4 + \lambda^2\}.$$

Theorem[Colin de Verdière, 83] N_{pp} is a constant for generic smooth metrics.

$\Longrightarrow$ Counting L^2 solutions to $(*)$ may not be very interesting.

But there is a continuum of non-L^2 solutions to $(*)$! How do we count the continuous spectrum ?

The spectral counting function

Remark For a compact manifold,

$$N_{pp}(\lambda) = \mathrm{Tr}\,\mathbf{1}(-\Delta \leqslant d^2/4 + \lambda^2).$$

The spectral counting function

Remark For a compact manifold,

$$N_{pp}(\lambda) = \mathrm{Tr}\, \mathbf{1}(-\Delta \leqslant d^2/4 + \lambda^2).$$

For a manifold with cusps, this trace is $+\infty$ for $\lambda > 0$.

The spectral counting function

Remark For a compact manifold,

$$N_{pp}(\lambda) = \mathrm{Tr}\,\mathbf{1}(-\Delta \leqslant d^2/4 + \lambda^2).$$

For a manifold with cusps, this trace is $+\infty$ for $\lambda > 0$. However, as $\tau \to \infty$,

$$\mathrm{Tr}\left[\mathbf{1}(y \leqslant e^\tau)\mathbf{1}(-\Delta \leqslant d^2/4 + \lambda^2)\right] = \kappa\tau\frac{\lambda}{\pi} + \Big[N_{pp}(\lambda) - S(\lambda) + C\Big] + o(1)$$

where S is a C^ω function.

The spectral counting function

Remark For a compact manifold,

$$N_{pp}(\lambda) = \mathrm{Tr}\, \mathbf{1}(-\Delta \leqslant d^2/4 + \lambda^2).$$

For a manifold with cusps, this trace is $+\infty$ for $\lambda > 0$. However, as $\tau \to \infty$,

$$\mathrm{Tr}\left[\mathbf{1}(y \leqslant e^\tau)\mathbf{1}(-\Delta \leqslant d^2/4 + \lambda^2)\right] = \kappa\tau\frac{\lambda}{\pi} + \left[N_{pp}(\lambda) - S(\lambda) + C\right] + o(1)$$

where S is a C^ω function. It is hence natural to define

$$N(\lambda) := N_{pp}(\lambda) - S(\lambda) + C.$$

S is the *scattering phase*, and it appears in the description of the projection on the subspace of $L^2(M)$ corresponding to the continuous spectrum.

Theorem[CdV 83, Müller, 86] We have the equivalent

$$N(\lambda) \sim \frac{\mathrm{vol}(B^*M)}{(2\pi)^{d+1}} \lambda^{d+1}.$$

Theorem[CdV 83, Müller, 86] We have the equivalent

$$N(\lambda) \sim \frac{\mathrm{vol}(B^*M)}{(2\pi)^{d+1}}\lambda^{d+1}.$$

Recall

Theorem[Levitan, Ivrii, Bérard, $\cdots$] If M is a compact manifold,

$$N(\lambda) = \frac{\mathrm{vol}(B^*M)}{(2\pi)^{d+1}}\lambda^{d+1} + R(\lambda),$$

where

1. $R(\lambda) = \mathcal{O}(\lambda^d)$.
2. If $|per| = 0$, $R(\lambda) = o(\lambda^d)$.
3. If $K \leqslant 0$, $R(\lambda) = \mathcal{O}(\lambda^d/\log\lambda)$.

Theorem[CdV 83, Müller, 86] We have the equivalent

$$N(\lambda) \sim \frac{\mathrm{vol}(B^*M)}{(2\pi)^{d+1}}\lambda^{d+1}.$$

We have

Theorem If M is a manifold with cusps,

$$N(\lambda) = \frac{\mathrm{vol}(B^*M)}{(2\pi)^{d+1}}\lambda^{d+1} - \frac{\kappa}{\pi}\lambda\log\lambda + \frac{\kappa(1-\log 2)}{\pi}\lambda + R(\lambda),$$

where

1. $R(\lambda) = \mathcal{O}(\lambda^d)$.
2. If $|per| = 0$, $R(\lambda) = o(\lambda^d)$.
3. If $K \leqslant 0$, $R(\lambda) = \mathcal{O}(\lambda^d/\log\lambda)$.

The terms in red only contribute when $d = 1$. Parnovski [95] $1.$ and $2.$ when $d = 1$; Selberg $3.$ when $d = 1$ and $K = -1$.

The resonances

The resonances are a discrete set of points in $\{\Re s < d/2\}$.
They are (by definition) the poles of the meromorphic continuation of $\phi(d/2 + i\lambda) = e^{2i\pi S(\lambda)}$.

The resonances

The resonances are a discrete set of points in $\{\Re s < d/2\}$.
They are (by definition) the poles of the meromorphic continuation of $\phi(d/2 + i\lambda) = e^{2i\pi S(\lambda)}$.
They satisfy:

$$-S'(\lambda) = Q(\lambda) + \frac{1}{2\pi} \sum_{\rho \text{ resonance}} \frac{d - 2\Re\rho}{|\rho - (d/2 + i\lambda)|^2},$$

where Q is a polynomial of order at most $2[d/2]$.

The resonances

The resonances are a discrete set of points in $\{\Re s < d/2\}$.
They are (by definition) the poles of the meromorphic continuation of $\phi(d/2 + i\lambda) = e^{2i\pi S(\lambda)}$.
They satisfy:

$$-S'(\lambda) = Q(\lambda) + \frac{1}{2\pi} \sum_{\rho \text{ resonance}} \frac{d - 2\Re\rho}{|\rho - (d/2 + i\lambda)|^2},$$

where Q is a polynomial of order at most $2\lfloor d/2 \rfloor$.
Each resonance ρ contributes by a Cauchy distribution centered at $\Im\rho$, and with width $d - 2\Re\rho$.
$\Longrightarrow -S$ is related to the counting function of the resonances.

First result for counting resonances

Theorem Assume $d \geqslant 2$. Then,

$$\#\{s \text{ resonance} \mid |s - d/2| \leqslant \lambda\} = -2S(\lambda) + R(\lambda),$$

where

1. $R(\lambda) = \mathcal{O}(\lambda^d)$.
2. If $|per| = 0$, $R(\lambda) = o(\lambda^d)$.

Remark. When $d = 1$, we only have $R(\lambda) = \mathcal{O}(\lambda^{3/2})$.

Second result on counting of resonances.

Theorem Let (M, g_0) be a manifold with cusps of constant curvature. Then for any metric g on M, sufficiently C^2-close to g_0, the following holds.

There exists $-\infty < \delta(g) < d/2$ such that

1. $\#\{\rho \text{ resonance} \mid \Re\rho < \delta(g),\ |\rho - d/2| \leqslant \lambda\} = \mathcal{O}(\lambda)$.
2. Let $B = \{\rho \text{ resonance} \mid \Re\rho > \delta(g)\}$. Then

$$\sum_{\rho\in B,\ |\Im\rho|\leqslant\lambda} d - 2\Re\rho = \frac{\kappa}{\pi}\lambda\log\lambda + C'\lambda + \mathcal{O}(\log\lambda), \qquad \text{(VM)}$$

$$\#\{\rho \in B,\ |\Im\rho| \leqslant \lambda\} = -2S(\lambda) + \frac{\ell_*}{\pi}\lambda + \mathcal{O}\left(\frac{\lambda^d}{\log\lambda}\right).$$

Remark. (VM) generalizes the Von Mangoldt formula. The constant ℓ_* has a geometric interpretation in terms of length of scattered geodesics.

$\psi \in \mathcal{S}(\mathbb{R})$, even and real

$$0\text{-Tr}\,\psi(\sqrt{-\Delta - d^2/4}) = \int_{\mathbb{R}} \hat{\psi}(t)\, 0\text{-Tr}\cos t\sqrt{-\Delta - d^2/4}dt$$

Kernel of $\cos t\sqrt{-\Delta - d^2/4}$:

$$K(x, x', t) \sim C_0 \sum_{k\geqslant 0} (-1/2)^k u_k(x, x') \sinh|t| \frac{(\cosh(d(x, x')) - \cosh t)^{k-d/2-1}}{\Gamma(k - d/2)}$$